INITIAL $L^2 \times \cdots \times L^2$ BOUNDS FOR MULTILINEAR OPERATORS

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ABSTRACT. The L^p boundedness theory of convolution operators is based on an initial $L^2 \to L^2$ estimate derived from the Fourier transform. The corresponding theory of multilinear operators lacks such a simple initial estimate in view of the unavailability of Plancherel's identity in this setting, and up to now it has not been clear what a natural initial estimate might be. In this work we obtain initial $L^2 \times \cdots \times L^2 \to L^{2/m}$ estimates for three types of important multilinear operators: rough singular integrals, multipliers of Hörmander type, and multipliers whose derivatives satisfy qualitative estimates. These estimates lay the foundation for the derivation of other L^p estimates for such operators.

1. INTRODUCTION AND PRELIMINARIES

The systematic study of multilinear operators in harmonic analysis was initiated by Coifman and Meyer in the seventies. Many important multilinear operators can be written as

$$T(f_1,\dots,f_m)(x) = W * (f_1 \otimes \cdots \otimes f_m)(x,\dots,x), \qquad x \in \mathbb{R}^n,$$

where f_j are Schwartz functions on $\mathbb{R}^n$, W is a tempered distribution on $(\mathbb{R}^n)^m$, and $(f_1 \otimes \cdots \otimes f_m)(x_1,\dots,x_m) = f_1(x_1)\cdots f_m(x_m)$. Alternatively $T(f_1,\dots,f_m)(x)$ can be expressed as

$$(1) \qquad \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)\widehat{W}(\xi_1,\dots,\xi_m)e^{2\pi i\langle x,\xi_1+\cdots+\xi_m\rangle}d\xi_1\cdots d\xi_n,$$

where $\widehat{f_j}(\xi) = \int_{\mathbb{R}^n} f_j(x)e^{-2\pi i\langle x,\xi\rangle}dx$ denotes the Fourier transform of f_j and $\widehat{W}$ is the distributional Fourier transform of W, which must be an L^∞ function if T is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some

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choice of indices that satisfy $1/p = 1/p_1 + \cdots + 1/p_m$. If such an estimate holds for T, then $\sigma := \widehat{W}$ is called a multilinear Fourier multiplier, and the operator in (1) is also denoted by $T_\sigma(f_1, \ldots, f_m)$. The first important result concerning multilinear Fourier multipliers is a nontrivial adaptation of Mihlin's multiplier condition, obtained by Coifman and Meyer [4]. The proof they gave is based on decomposing the multiplier as a sum of products of linear operators via Littlewood-Paley and Fourier series expansions. This powerful idea has essentially been the only technique available in this area until the appearance of the wave-packet decompositions in the work of Lacey and Thiele [28, 29] on the bilinear Hilbert transform.

If the distribution W has the form

$$W = \text{p.v.}\ \frac{1}{|(y_1, \ldots, y_m)|^{mn}} \Omega\Big(\frac{(y_1, \ldots, y_m)}{|(y_1, \ldots, y_m)|}\Big)$$

for some integrable function Ω on the sphere $\mathbb{S}^{mn-1}$ with integral zero, then T is called an m-linear homogeneous singular. The associated operator is bounded if Ω is smooth but it could be unbounded if Ω is merely integrable; see [20]. In this paper we focus on the intermediate situation where Ω lies in L^q for some $q \in (1, \infty]$; these Ω's give rise to rough m-linear homogeneous singular integrals. The study of bilinear homogeneous singular integrals was initiated by Coifman and Meyer in [5] who addressed the case where Ω is a function of bounded variation. The boundedness of T in the more difficult case when Ω is merely in L^∞ was not solved until four decades later in [17] in terms of wavelet decompositions. Prior to that, the first author and Torres [23], [24] had proved boundedness for T for any m when Ω satisfies a Lipschitz condition on the sphere. In the case $m = 1$ the known results are much better. For instance, Calderón and Zygmund [3] showed that T is bounded in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if $\Omega \in L\log L(\mathbb{S}^{n-1})$. This result was improved by Coifman and Weiss [6] under the less restrictive condition that Ω belongs to the Hardy space $H^1(\mathbb{S}^{n-1})$.

One fundamental difference between linear convolution operators and multilinear convolution operators of type (1) is an initial estimate. In the linear case the initial estimate is usually $L^2 \to L^2$ and this is obtained by Plancherel's identity. There is not a straightforward initial estimate for multilinear operators and in most times, it is difficult to find one. Inspired by [17], the first two authors and Slavíková [19] obtained a bilinear substitute of the Plancherel criterion for $L^2 \times L^2 \to L^1$ boundedness for multipliers in $L^q(\mathbb{R}^n)$ $(0 < q < 4)$ with sufficiently many bounded derivatives. This result has also been proved by Kato, Miyachi, and Tomita [26] and has found many applications; see for instance [19, 33].

Overcoming the combinatorial complexity that arises from multilinearity, in this paper we develop a method that yields the crucial

$$\overbrace{L^2(\mathbb{R}^n)\times\cdots\times L^2(\mathbb{R}^n)}^{m \text{ times}} \to L^{2/m}(\mathbb{R}^n)$$

estimates for a variety of m-linear operators, including rough homogeneous singular integrals and multipliers. Our results contribute to the recent surge of activity in the theory of rough multilinear singular integrals, see for instance [7, 10, 11, 19, 25, 26].

We first present a sharp $L^2\times\cdots\times L^2 \to L^{2/m}$ boundedness criterion for a multiplier with bounded derivatives up to a certain order. This provides a multilinear extension of the main result in [19].

Theorem 1.1. *Let m be a positive integer with $m\geq 2$ and $1<q<\frac{2m}{m-1}$. Set M_q to be a positive integer satisfying*

$$M_q > \frac{m(m-1)n}{2m-(m-1)q}.$$

Suppose that $\sigma\in L^q((\mathbb{R}^n)^m)\bigcap \mathscr{C}^{M_q}((\mathbb{R}^n)^m)$ with

$$\big\|\partial^\alpha \sigma\big\|_{L^\infty((\mathbb{R}^n)^m)} \leq D_0, \quad \text{for } |\alpha|\leq M_q. \tag{2}$$

Then the estimate

$$\big\|T_\sigma(f_1,\dots,f_m)\big\|_{L^{2/m}(\mathbb{R}^n)} \lesssim D_0^{1-\frac{(m-1)q}{2m}} \Big(\|\sigma\|_{L^q((\mathbb{R}^n)^m)}\Big)^{\frac{(m-1)q}{2m}} \prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

is valid for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

Remark 1. (i) The condition $1<q<\frac{2m}{m-1}$ can be relaxed to $0<q<\frac{2m}{m-1}$ as a multiplier $\sigma\in L^q$ with $q\in(0,\frac{2m}{m-1})$ is also in L^q with $q\in(1,\frac{2m}{m-1})$ by condition (2) with $\alpha=0$.

(ii) The number of derivatives M_q is sharp. One can verify this by modifying the example in [19, Section 3].

Next we discuss multilinear rough singular integral operators. For a fixed function Ω on the unit sphere $\mathbb{S}^{mn-1}$ and for $\vec{y}':=\vec{y}/|\vec{y}|\in\mathbb{S}^{mn-1}$ we let

$$K(\vec{y}) := \frac{\Omega(\vec{y}')}{|\vec{y}|^{mn}}. \tag{3}$$

We then define the corresponding multilinear operator

$$\mathcal{L}_\Omega\big(f_1,\dots,f_m\big)(x) := p.v.\int_{(\mathbb{R}^n)^m} K(\vec{y})\prod_{j=1}^m f_j(x-y_j)\, d\vec{y}, \quad x\in\mathbb{R}^n$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

Theorem 1.2. *Suppose that $\frac{2m}{m+1} < q \le \infty$ and let $\Omega \in L^q(\mathbb{S}^{mn-1})$ satisfying $\int_{\mathbb{S}^{mn-1}} \Omega \, d\sigma_{mn-1} = 0$. Then there exists a constant $C = C_{n,m,q} > 0$ such that*

$$\big\|\mathcal{L}_\Omega(f_1,\dots,f_m)\big\|_{L^{2/m}(\mathbb{R}^n)} \le C\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

The last result of this paper is about the boundedness of multilinear multiplier operators of Hörmander type. The multilinear multiplier operator associated with a bounded function σ on $(\mathbb{R}^n)^m$ is defined as in (8);

$$T_\sigma\big(f_1,\dots,f_m\big)(x) := \int_{(\mathbb{R}^n)^m} \sigma(\vec{\xi}\,)\Big(\prod_{j=1}^m \widehat{f}_j(\xi_j)\Big)e^{2\pi i\langle x,\sum_{j=1}^m \xi_j\rangle}d\vec{\xi}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$. We choose a Schwartz function $\Phi^{(m)}$ on $(\mathbb{R}^n)^m$ having the properties that $\widehat{\Phi^{(m)}}$ is positive and supported in the annulus $\{\vec{\xi} \in (\mathbb{R}^n)^m : 1/2 \le |\vec{\xi}\,| \le 2\}$, and $\sum_{\gamma\in\mathbb{Z}} \widehat{\Phi^{(m)}}(\vec{\xi}/2^\gamma) = 1$ for $\vec{\xi} \ne \vec{0}$. In the linear case $m = 1$, under the assumption

$$\sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\cdot)\widehat{\Phi^{(1)}}\big\|_{L^q_s(\mathbb{R}^n)} < \infty,$$

the condition

$$s > \max\big(|n/2-n/p|, n/q\big)$$

implies the boundedness of T_σ from $L^p(\mathbb{R}^n)$ to itself. Recently, the bilinear analogue of this result was obtained in the series of papers [18, 21, 32] by Grafakos, He, Honzík, Miyachi, Nguyen, and Tomita; all of these results were inspired by the fundamental work of Tomita [34] in this direction.

Theorem 1.3. *Let $1 < q < \infty$ and*

$$s > \max\big((m-1)n/2, mn/q\big). \tag{4}$$

Then there exists an absolute constant $C = C_{n,m,q,s} > 0$ such that

$$\|T_\sigma(f_1,\dots,f_m)\|_{L^{2/m}(\mathbb{R}^n)} \le C\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\,\vec{\cdot}\,)\widehat{\Phi^{(m)}}\big\|_{L^q_s((\mathbb{R}^n)^m)}\prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

We remark that for $1 < q \le 2$ this result has been obtained by [34] and [22], so Theorem 1.3 is new only in the case $q > 2$; this corresponds to the classical result of Calderón and Torchinsky [2] in the linear setting. The sharpness of condition (4) was addressed in [18, Theorem 2].

We design two novel ideas to deal with the above results: (I) An innovative decomposition of an m-linear multiplier into sums of products so that l-linear Plancherel type estimates ($1 \le l \le m$) can be used; see Proposition 2.4. (II) An effective way to split lattice points in $(\mathbb{Z}^n)^m$ *into groups of columns* for the purposes of obtaining $L^2 \times \cdots \times L^2 \to L^{2/m}$ estimates; for details see Section 3.

It is inevitable to introduce complicated notation in order to comprehensively present the proofs in the general framework of m-linear operators; for this reason we urge the readers to restrict attention to the case $m = 3$, which already presents several new ingredients compared with the case $m = 2$ and contains the main ideas.

Notation. C will denote inessential constants that may vary from occurrence to occurrence. $A \lesssim B$ means $A \le CB$ with C independent of A and B, and write $A \approx B$ if both $A \lesssim B$ and $B \lesssim A$ hold. We denote the set of natural numbers by $\mathbb{N}$ and of integers by $\mathbb{Z}$; we also set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout this paper, the index $m \in \mathbb{N}$ will be the degree of multilinearity of operators.

2. Plancherel type estimates

Let ω be a compactly supported function defined on $\mathbb{R}^n$ which is smooth up to order M_1, where $M_1 \ge n+1$ is an integer. For $\lambda \in \mathbb{N}_0$ let $\{\omega_k^\lambda\}_{k\in\mathbb{Z}^n}$ be a sequence of compactly supported functions defined on $\mathbb{R}^n$ by the formula $\omega_k^\lambda(\xi) = 2^{\lambda n/2}\omega(2^\lambda \xi - k)$.It is easy to see that ω_k^λ is supported in the ball $B(2^{-\lambda}k, C2^{-\lambda})$ of radius $C2^{-\lambda}$, centered at $2^{-\lambda}k$, for some fixed C. This leads to the following properties:

(i) $\{\omega_k^\lambda\}_{k\in\mathbb{Z}^n}$ have almost disjoint supports.

(ii) $\sum_{k\in\mathbb{Z}^n} |\omega_k^\lambda(\xi)| \le 2^{\lambda n/2}$ for all $\xi \in \mathbb{R}^n$.

As a consequence of (i) and (ii) we obtain

$$(5) \qquad \Big(\sum_{k\in\mathbb{Z}^n} \big|\omega_k^\lambda(\xi)\big|^q\Big)^{1/q} \approx_q \sum_{k\in\mathbb{Z}^n} \big|\omega_k^\lambda(\xi)\big| \le 2^{\lambda n/2}$$

for any $0 < q < \infty$, due to the property of the supports. We define

$$(6) \qquad \omega_{\vec{k}}^\lambda(\vec{\xi}) := \omega_{k_1}^\lambda(\xi_1)\cdots\omega_{k_m}^\lambda(\xi_m)$$

where $\vec{k} := (k_1,\dots,k_m) \in (\mathbb{Z}^n)^m$ and $\vec{\xi} = (\xi_1,\dots,\xi_m) \in (\mathbb{R}^n)^m$. Let $\mathcal{U}$ be a subset of $(\mathbb{Z}^n)^m$ and $\{b_{\vec{k}}^\lambda\}_{\vec{k}\in\mathcal{U}}$ be a sequence of complex numbers. We define

$$(7) \qquad \sigma^\lambda(\vec{\xi}) := \sum_{\vec{k}\in\mathcal{U}} b_{\vec{k}}^\lambda \omega_{\vec{k}}^\lambda(\vec{\xi})$$

and the corresponding m-linear multiplier operator by

$$T_{\sigma^\lambda}\big(f_1,\dots,f_m\big)(x) := \int_{(\mathbb{R}^n)^m} \sigma^\lambda(\vec{\xi}\,)\Big(\prod_{j=1}^m \widehat{f_j}(\xi_j)\Big)e^{2\pi i\langle x,\sum_{j=1}^m \xi_j\rangle}d\vec{\xi} \tag{8}$$

for $x\in\mathbb{R}^n$ and Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$. This operator coincides with that in (1) when $\sigma^\lambda = \widehat{W}$.

The multiplier σ^λ defined in (7) appears naturally in the decomposition of many operators and plays a key role in the understanding of their boundedness. Actually in the bilinear case, one has the estimate ([1, 19])

$$\|T_{\sigma^\lambda}(f,g)\|_{L^1(\mathbb{R}^n)} \le C\|\{b^\lambda_{\vec{k}}\}\|_{\ell^\infty}2^{\lambda n}|\mathcal{U}|^{1/4}\|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}. \tag{9}$$

The presence of $\|\{b^\lambda_{\vec{k}}\}\|_{\ell^\infty}$ and $|U|^{1/4}$ indicate the contribution of both the height and the support of σ^λ. This phenomenon is dissimilar to the L^2 boundedness of linear multipliers, where the support of the multiplier plays no role. Motivated by many applications in which σ^λ is an important building block, in this work we obtain the m-linear version of (9).

Proposition 2.1. *Let N be a positive integer and $\mathcal{U}$ be a subset of $(\mathbb{Z}^n)^m$ with $|\mathcal{U}|\le N$. For $\lambda\ge 0$, let $\{b^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}$ be a sequence of complex numbers satisfying $\|\{b^\lambda_{\vec{k}}\}_{\vec{k}}\|_{\ell^\infty}\le A_\lambda$. Let σ^λ be defined as in (7). Then there exists a constant $C = C_{n,m} > 0$ such that*

$$\big\|T_{\sigma^\lambda}\big(f_1,\dots,f_m\big)\big\|_{L^{2/m}(\mathbb{R}^n)} \le CA_\lambda N^{\frac{m-1}{2m}}2^{\frac{\lambda mn}{2}}\prod_{j=1}^m \|f_j\|_{L^2(\mathbb{R}^n)}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

When $m = 1$, Proposition 2.1 follows from Plancherel's identity and yields a bound on the L^2 norm of the corresponding linear operator T_σ that depends only on the height of the multiplier σ. For $m = 2$, it coincides with (9). Below we focus on the consequences of Proposition 2.1 while its proof is postponed until the next section.

Remark 2. After completing this paper, we were informed that Kato, Miyachi, and Tomita [27] recently obtained a result that implies Proposition 2.1. Their proof is independent of ours and builds on their previous work in [26].

The restriction $|\mathcal{U}|\le N$ in Proposition 2.1 can be interpreted in terms of the compact support condition of σ^λ. Indeed, the support of σ^λ has measure bounded by a constant times $N2^{-\lambda mn}$.

As we have seen in the proof of Proposition 2.1, the $L^2\times\cdots\times L^2\to L^{2/m}$ boundedness of m-linear multiplier operator T_σ, $m\ge 2$, may be affected by the support of σ while the L^2 boundedness depends only on $\|\sigma\|_{L^\infty}$ in the linear setting.

On the other hand, the following "support-independent" result could be obtained from Proposition 2.1 under an extra ℓ^q condition which is satisfied in many applications.

Proposition 2.2. *Let $m \in \mathbb{N}$ and $0 < q < \frac{2m}{m-1}$. Fixing $\lambda \in \mathbb{N}_0$, let $\{\omega^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}$ be wavelets of level λ. Suppose $\{b^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}$ is a sequence of complex numbers satisfying $\|\{b^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\|_{\ell^\infty} \le A_\lambda$ and $\|\{b^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\|_{\ell^q} \le B_{\lambda,q}$. Then the m-linear multiplier σ^λ, defined in (7) with $\mathcal{U} = (\mathbb{Z}^n)^m$, satisfies*

$$\big\|T_{\sigma^\lambda}(f_1,\dots,f_m)\big\|_{L^{2/m}} \lesssim A_\lambda^{1-\frac{(m-1)q}{2m}} B_{\lambda,q}^{\frac{(m-1)q}{2m}} 2^{\lambda mn/2} \prod_{j=1}^m \|f_j\|_{L^2}$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$.

Proof. When $m = 1$, it is clear from Plancherel's identity. Therefore we assume $m \ge 2$.

For $r \in \mathbb{N}$ let

$$\mathcal{U}^\lambda_r := \big\{\vec{k} \in (\mathbb{Z}^n)^m : A2^{-r} < |b^\lambda_{\vec{k}}| \le A2^{-r+1}\big\}.$$

As $\|\{b^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\|_{\ell^\infty} \le A_\lambda$, $(\mathbb{Z}^n)^m$ can be written as the disjoint union of $\mathcal{U}^\lambda_r$, $r \in \mathbb{N}$, and thus we may decompose σ^λ as

$$\sigma^\lambda = \sum_{r\in\mathbb{N}} \sigma^\lambda_r$$

where $\sigma^\lambda_r := \sum_{\vec{k}\in\mathcal{U}^\lambda_r} b^\lambda_{\vec{k}} \omega^\lambda_{\vec{k}}$. Observe that

$$2^{-r}A_\lambda |\mathcal{U}^\lambda_r|^{1/q} \le \Big(\sum_{\vec{k}\in\mathcal{U}^\lambda_r} |b^\lambda_{\vec{k}}|^q\Big)^{1/q} \le B_{\lambda,q}, \tag{10}$$

which implies

$$|\mathcal{U}^\lambda_r| \le \Big(\frac{B_{\lambda,q}}{2^{-r}A_\lambda}\Big)^q. \tag{11}$$

Applying Proposition 2.1 and (11) to each σ^λ_r, we obtain

$$\begin{aligned}\big\|T_{\sigma^\lambda_r}(f_1,\dots,f_m)\big\|_{L^{2/m}} &\lesssim |\mathcal{U}^\lambda_r|^{(m-1)/2m} 2^{\lambda mn/2}(A_\lambda 2^{-r}) \prod_{j=1}^m \|f_j\|_{L^2} \\ &\le (A_\lambda 2^{-r})^{1-\frac{(m-1)q}{2m}} B_{\lambda,q}^{\frac{q(m-1)}{2m}} 2^{\lambda mn/2} \prod_{j=1}^m \|f_j\|_{L^2}.\end{aligned}$$

Taking $\ell^{2/m}$-norm over $r \in \mathbb{N}$, we have

$$\big\|T_{\sigma^\lambda}(f_1,\dots,f_m)\big\|_{L^{2/m}} \le \Big(\sum_{r\in\mathbb{N}} \big\|T_{\sigma^\lambda_r}(f_1,\dots,f_m)\big\|_{L^{2/m}}^{2/m}\Big)^{m/2}$$

$$\lesssim A_\lambda^{1-\frac{(m-1)q}{2m}} B_{\lambda,q}^{\frac{(m-1)q}{2m}} 2^{\lambda mn/2} \|f_1\|_{L^2} \cdots \|f_m\|_{L^2},$$

since $1 - \frac{(m-1)q}{2m} > 0$ and $2/m \leq 1$. $\square$

For the case $m \geq 2$ and $q \geq \frac{2m}{m-1}$, we have the following substitute result under the extra condition that all $\vec{k}$ belong to a ball of radius $C2^\lambda$, centered at the origin, which means that σ^λ is contained in a ball of radius comparable to 1.

Proposition 2.3. *Let m be a positive integer with $m \geq 2$ and $\frac{2m}{m-1} \leq q < \infty$. For each $\lambda \in \mathbb{N}_0$ let $\{\omega_{\vec{k}}^\lambda\}$ be wavelets of level λ. Let $\mathcal{U}^\lambda := \{\vec{k} \in (\mathbb{Z}^n)^m : |\vec{k}| \leq C2^\lambda\}$ for some $C > 0$. Suppose $\{b_{\vec{k}}^\lambda\}_{\vec{k}\in(\mathbb{Z}^n)^m}$ is a sequence of complex numbers with $\|\{b_{\vec{k}}^\lambda\}_{\vec{k}\in(\mathbb{Z}^n)^m}\|_{\ell^q} \leq B_{\lambda,q}$. Then the m-linear multiplier σ^λ, defined in (7) with $\mathcal{U} = \mathcal{U}^\lambda$, satisfies*

$$\big\|T_{\sigma^\lambda}(f_1,\dots,f_m)\big\|_{L^{2/m}} \lesssim B_{\lambda,q} D_{\lambda,q,m} \Big(\prod_{j=1}^m \|f_j\|_{L^2}\Big)$$

for Schwartz functions $f_1,\dots,f_m$ on $\mathbb{R}^n$, where

$$D_{\lambda,q,m} := \begin{cases} \lambda^{m/2} 2^{\lambda mn/2}, & q = \frac{2m}{m-1} \\ 2^{\lambda n(\frac{2m-1}{2} - \frac{m}{q})}, & q > \frac{2m}{m-1}. \end{cases} \tag{12}$$

Proof. Pick $r_{max} \in \mathbb{N}$ satisfying $\frac{\lambda mn}{q} \leq r_{max} < \frac{\lambda mn}{q} + 1$. Define

$$\mathcal{U}_{r_{max}}^\lambda := \{\vec{k} \in \mathcal{U}^\lambda : |b_{\vec{k}}^\lambda| \leq 2^{-r_{max}+1} B_{\lambda,q}\}$$

and for $1 \leq r < r_{max}$

$$\mathcal{U}_r^\lambda := \{\vec{k} \in \mathcal{U}^\lambda : 2^{-r} B_{\lambda,q} < |b_{\vec{k}}^\lambda| \leq 2^{-r+1} B_{\lambda,q}\}.$$

Since $|b_{\vec{k}}^\lambda| \leq B_{\lambda,q}$ for all $\vec{k} \in (\mathbb{Z}^n)^m$, we can write

$$\sigma^\lambda = \sum_{r=1}^{r_{max}} \sigma_r^\lambda$$

where $\sigma_r^\lambda := \sum_{\vec{k}\in\mathcal{U}_r^\lambda} b_{\vec{k}}^\lambda \omega_{\vec{k}}^\lambda$. Using the same argument in (10), we see that

$$|\mathcal{U}_r^\lambda| \leq 2^{rq}, \qquad 1 \leq r < r_{max} \tag{13}$$

and

$$|\mathcal{U}_{r_{max}}^\lambda| \leq |\mathcal{U}^\lambda| \lesssim 2^{\lambda mn} \leq 2^{r_{max} q}. \tag{14}$$

Applying Proposition 2.1, (13), and (14) to each σ_r^λ, we obtain

$$\begin{aligned}\big\|T_{\sigma_r^\lambda}(f_1,\dots,f_m)\big\|_{L^{2/m}} &\lesssim |\mathcal{U}_r^\lambda|^{(m-1)/2m}2^{\lambda mn/2}\|\{b_{\vec{k}}^\lambda\}_{\vec{k}\in\mathcal{U}_r^\lambda}\|_{\ell^\infty}\prod_{j=1}^m\|f_j\|_{L^2}\\ &\lesssim 2^{rq(m-1)/2m}2^{\lambda mn/2}2^{-r}B_{\lambda,q}\prod_{j=1}^m\|f_j\|_{L^2}\\ &= B_{\lambda,q}2^{\lambda mn/2}2^{r(\frac{q(m-1)}{2m}-1)}\prod_{j=1}^m\|f_j\|_{L^2}.\end{aligned}$$

Taking the $\ell^{2/m}$ quasi-norm over $1\le r\le r_{max}$, we have

$$\begin{aligned}\|T_{\sigma^\lambda}(f_1,\dots,f_m)\|_{L^{2/m}} &\le \Big(\sum_{r=1}^{r_{max}}\big\|T_{\sigma_r^\lambda}(f_1,\dots,f_m)\big\|_{L^{2/m}}^{2/m}\Big)^{m/2}\\ &\lesssim B_{\lambda,q}2^{\lambda mn/2}\Big(\sum_{r=1}^{r_{max}}2^{r(\frac{q(m-1)}{2m}-1)\frac{2}{m}}\Big)^{m/2}\prod_{j=1}^m\|f_j\|_{L^2}.\end{aligned}$$

We note that

$$\begin{aligned}\Big(\sum_{r=1}^{r_{max}}2^{r(\frac{q(m-1)}{2m}-1)\frac{2}{m}}\Big)^{m/2} &\approx \begin{cases} r_{max}^{m/2}, & q=\frac{2m}{m-1}\\ 2^{r_{max}(\frac{q(m-1)}{2m}-1)}, & q>\frac{2m}{m-1}\end{cases}\\ &\lesssim \begin{cases}\lambda^{m/2}, & q=\frac{2m}{m-1}\\ 2^{\lambda n(\frac{m-1}{2}-\frac{m}{q})}, & q>\frac{2m}{m-1},\end{cases}\end{aligned}$$

which completes the proof. □

In the study of bilinear rough singular integrals and bilinear Hörmander multipliers, an argument splitting the problem to diagonal and off-diagonal cases is utilized. The off-diagonal case uses Plancherel's theorem and a pointwise control. In the diagonal case, we employ the bilinear result of Plancherel type, which is actually the driving force of this work. We now present a multilinear version generalizing and combining these two parts, which shows that all l-linear Plancherel type result, $1\le l\le m$, is necessary in the study of many m-linear multipliers.

For $\mu\in\mathbb{N}_0$ let $\mathcal{V}^\mu$ be a subset of $\{\vec{k}\in(\mathbb{Z}^n)^m : 2^{\mu-c_0}\le|\vec{k}|\le 2^{\mu+c_0}\}$ for some $c_0\ge 1$. Let M be a positive constant and for each $1\le l\le m$ let

$$\mathcal{V}_l^\mu := \big\{\vec{k}\in\mathcal{V}^\mu : |k_1|,\dots,|k_l|\ge M>|k_{l+1}|,\dots,|k_m|\big\}.$$

We also define $L_k^\lambda f := \big(\omega_k^\lambda\widehat{f}\big)^\vee$ and $L_k^{\lambda,\gamma}f := \big(\omega_k^\lambda(\cdot/2^\gamma)\widehat{f}\big)^\vee$ for $k\in\mathbb{Z}^n$.

Proposition 2.4. *Let m be a positive integer with $m\ge 2$ and $0<q<\infty$. For each $\lambda\in\mathbb{N}_0$, let $\{\omega_{\vec{k}}^\lambda\}_{\vec{k}}$ be wavelets of level λ. Suppose that*

$\{b_{\vec{k}}^{\lambda,\gamma,\mu}\}_{\gamma,\mu\in\mathbb{Z},\vec{k}\in(\mathbb{Z}^n)^m}$ *is a sequence of complex numbers satisfying*

$$\sup_{\gamma\in\mathbb{Z}}\big\|\{b_{\vec{k}}^{\lambda,\gamma,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^\infty}\le A_{\lambda,\mu}$$

and

$$\sup_{\gamma\in\mathbb{Z}}\big\|\{b_{\vec{k}}^{\lambda,\gamma,\mu}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^q}\le B_{\lambda,\mu,q}.$$

Then the following statements hold:

(1) For $1\le r\le 2$ there exists a constant $C>0$, independent of λ,μ, such that

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{k}\in\mathcal{V}_1^{\lambda+\mu}}b_{\vec{k}}^{\lambda,\gamma,\mu}L_{k_1}^{\lambda,\gamma}f_1^{\lambda,\gamma,\mu}\prod_{j=2}^m L_{k_j}^{\lambda,\gamma}f_j\Big|^r\Big)^{1/r}\Big\|_{L^{2/m}}$$

$$\le CA_{\lambda,\mu}2^{\lambda mn/2}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_1^{\lambda,\gamma,\mu}\|_{L^2}^r\Big)^{1/r}\prod_{i=2}^m\|f_i\|_{L^2}$$

for Schwartz functions $f_2,\dots,f_m$ on $\mathbb{R}^n$ and a sequence of Schwartz functions $\{f_1^{\lambda,\gamma,\mu}\}_{\lambda,\gamma,\mu}$ on $\mathbb{R}^n$.

(2) For $2\le l\le m$ and $0<q<\frac{2l}{l-1}$, there exists a constant $C>0$, independent of λ,μ, such that

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{k}\in\mathcal{V}_l^{\lambda+\mu}}b_{\vec{k}}^{\lambda,\gamma,\mu}\Big(\prod_{j=1}^l L_{k_j}^{\lambda,\gamma}f_j^{\lambda,\gamma,\mu}\Big)\Big(\prod_{j=l+1}^m L_{k_j}^{\lambda,\gamma}f_j\Big)\Big|\Big\|_{L^{2/m}}$$

$$\le CA_{\lambda,\mu}^{1-\frac{(l-1)q}{2l}}B_{\lambda,\mu,q}^{\frac{(l-1)q}{2l}}2^{\lambda mn/2}\Big[\prod_{j=1}^l\Big(\sum_{\gamma\in\mathbb{Z}}\|f_j^{\lambda,\gamma,\mu}\|_{L^2}^2\Big)^{1/2}\Big]\Big[\prod_{j=l+1}^m\|f_j\|_{L^2}\Big]$$

for Schwartz functions $f_{l+1},\dots,f_m$ on $\mathbb{R}^n$ and sequences of Schwartz functions $\{f_j^{\lambda,\gamma,\mu}\}_{\lambda,\gamma,\mu}$, $j=1,\dots,l$, where $\prod_{m+1}^m$ is understood as empty.

(3) For $2\le l\le m$ and $\frac{2l}{l-1}\le q<\infty$, there exists a constant $C>0$, independent of λ, such that

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{k}\in\mathcal{V}_l^{\lambda}}b_{\vec{k}}^{\lambda,\gamma}\Big(\prod_{j=1}^l L_{k_j}^{\lambda,\gamma}f_j^{\lambda,\gamma}\Big)\Big(\prod_{j=l+1}^m L_{k_j}^{\lambda,\gamma}f_j\Big)\Big|\Big\|_{L^{2/m}}$$

$$\le CB_{\lambda,q}D_{\lambda,q,l}2^{\lambda(m-l)n/2}\Big[\prod_{j=1}^l\Big(\sum_{\gamma\in\mathbb{Z}}\|f_j^{\lambda,\gamma}\|_{L^2}^2\Big)^{1/2}\Big]\Big[\prod_{j=l+1}^m\|f_j\|_{L^2}\Big]$$

for Schwartz functions $f_{l+1},\dots,f_m$ on $\mathbb{R}^n$ and sequences of Schwartz functions $\{f_j^{\lambda,\gamma}\}_{\lambda,\gamma}$, $j=1,\dots,l$, where $b_{\vec{k}}^{\lambda,\gamma}:=b_{\vec{k}}^{\lambda,\gamma,0}$, $B_{\lambda,q}:=B_{\lambda,0,q}$, and $D_{\lambda,q,l}$ is defined as in (12).

The proof of Proposition 2.4 is given in the next section following that of Proposition 2.1.

3. Proofs of Proposition 2.1 and Proposition 2.4

When $m = 1$, Proposition 2.1 follows immediately from Plancherel's identity. Thus, we will be concerned only with the case $m \geq 2$. For the bilinear case $m = 2$, a concept called column is used; see, for instance, [1, 19]. Let $\mathcal{U}$ be any subset of $\mathbb{Z}^n \times \mathbb{Z}^n$. For any $(k_1, k_2) \in \mathcal{U}$, a column $Col^{\mathcal{U}}_{k_1}$ is defined as the subset of $\mathbb{Z}^n$ consisting of k for which $(k_1, k) \in \mathcal{U}$. Similarly, $Col^{\mathcal{U}}_{k_2}$ means the set of $k \in \mathbb{Z}^n$ satisfying $(k, k_2) \in \mathcal{U}$. We generalize the concept of columns to the multilinear case. For a fixed $\vec{k} \in (\mathbb{Z}^n)^m$, $1 \leq l \leq m$, and $1 \leq j_1 \leq \cdots \leq j_l \leq m$ let

$$\vec{k}^{j_1,\ldots,j_l} := (k_{j_1}, \ldots, k_{j_l})$$

denote the vector in $(\mathbb{Z}^n)^l$ consisting of $j_1, \ldots, j_l$ components of $\vec{k}$ and $\vec{k}^{*j_1,j_2,\ldots,j_l}$ stand for the vector in $(\mathbb{Z}^n)^{m-l}$, consisting of $\vec{k}$ excepting $j_1, \ldots, j_l$ components (e.g. $\vec{k}^{*1,\ldots,j} = \vec{k}^{j+1,\ldots,m} = (k_{j+1}, \ldots, k_m) \in (\mathbb{Z}^n)^{m-j}$). For any sets $\mathcal{U}$ in $(\mathbb{Z}^n)^m$, $1 \leq j \leq m$, and $1 \leq j_1 \leq \cdots \leq j_l \leq m$ let

$$\mathcal{P}_j\mathcal{U} := \left\{ k_j \in \mathbb{Z}^n : \vec{k} \in \mathcal{U} \ \text{ for some } \vec{k}^{*j} \in (\mathbb{Z}^n)^{m-1} \right\}$$

$$\mathcal{P}_{*j_1,\ldots,j_l}\mathcal{U} := \left\{ \vec{k}^{*j_1,\ldots,j_l} \in (\mathbb{Z}^n)^{m-l} : \vec{k} \in \mathcal{U} \ \text{ for some } k_{j_1}, \ldots, k_{j_l} \in \mathbb{Z}^n \right\}$$

be the projections of $\mathcal{U}$ onto the k_j-column and $\vec{k}^{*j_1,\ldots,j_l}$-plane, respectively. For a fixed $\vec{k}^{*j_1,\ldots,j_l} \in \mathcal{P}_{*j_1,\ldots,j_l}\mathcal{U}$, we define

$$Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_l}} := \{ \vec{k}^{j_1,\ldots,j_l} \in (\mathbb{Z}^n)^l : \vec{k} = (k_1, \ldots, k_m) \in \mathcal{U} \}.$$

Then we observe that

$$\sum_{\vec{k} \in \mathcal{U}} \cdots = \sum_{\vec{k}^{*j_1,\ldots,j_l} \in \mathcal{P}_{*j_1,\ldots,j_l}\mathcal{U}} \Bigg(\sum_{\vec{k}^{j_1,\ldots,j_l} \in Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_l}}} \cdots \Bigg). \tag{15}$$

Furthermore, for each $\vec{k}^{*j_1,\ldots,j_{l-1}}$ we have

$$Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_l}} = \bigcup_{k_{j_l} \in \mathcal{P}_{j_l} Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_l}}} Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_{l-1}}} \times \{k_{j_l}\}$$

and this allows us to wrtie

$$\sum_{\vec{k} \in \mathcal{U}} \cdots = \sum_{\vec{k}^{*j_1,\ldots,j_l} \in \mathcal{P}_{*j_1,\ldots,j_l}\mathcal{U}} \Bigg(\sum_{k_{j_l} \in \mathcal{P}_{j_l} Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_l}}} \Bigg(\sum_{\vec{k}^{j_1,\ldots,j_{l-1}} \in Col^{\mathcal{U}}_{\vec{k}^{*j_1,\ldots,j_{l-1}}}} \cdots \Bigg)\Bigg). \tag{16}$$

To eliminate possible ambiguities, we clarify that $Col^{\mathcal{U}}_{\vec{k}^{*j_1,\dots,j_l}}$ consists of points in $(\mathbb{Z}^n)^l$ denoted by $(k_{j_1},k_{j_2},\dots,k_{j_l})$, and $\mathcal{P}_{j_l}(k_{j_1},k_{j_2},\dots,k_{j_l})=k_{j_l}$; here $\mathcal{P}_{j_l}$ is used slightly differently than previously defined.

To make it easier to understand, let us think about the case $n=1$ and $m=3$. $\mathcal{P}_1\mathcal{U}$ is the projection of $\mathcal{U}$ to the first coordinate. $\mathcal{P}_{*1}\mathcal{U}$ is the projection of $\mathcal{U}$ to the (k_2,k_3)-plane. $Col^{\mathcal{U}}_{\vec{k}^{*1}}$ is an 1-column in $\mathbb{Z}^3\cap\mathcal{U}$ with $(k_2,k_3)=\vec{k}^{*1}$ fixed. When $j_l=1$, identity (15) says that

$$\sum_{\vec{k}\in\mathcal{U}}\cdots=\sum_{(k_2,k_3)\in\mathcal{P}_{*1}\mathcal{U}}\Big(\sum_{k_1\in Col^{\mathcal{U}}_{k_2,k_3}}\cdots\Big).$$

When $(j_1,\dots,j_l)=(1,2)$, identity (16) says that

$$\sum_{\vec{k}\in\mathcal{U}}\cdots=\sum_{k_3\in\mathcal{P}_3\mathcal{U}}\Big(\sum_{k_2\in\mathcal{P}_2 Col^{\mathcal{U}}_{k_3}}\Big(\sum_{k_1\in Col^{\mathcal{U}}_{k_2,k_3}}\cdots\Big)\Big).$$

The proof of Proposition 2.1 is based on the decompositions in (15) and (16) and on the following lemma.

Lemma 3.1. *Let $m\ge 2$ and $\mathcal{U}$ be a subset of $(\mathbb{Z}^n)^m$. Let $\lambda\in\mathbb{N}_0$ and $\{\omega^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}$ be wavelets whose level is λ. Let $\sigma^\lambda=\sum_{\vec{k}\in\mathcal{U}}b^\lambda_{\vec{k}}\omega^\lambda_{\vec{k}}$, where $\{b^\lambda_{\vec{k}}\}_{\vec{k}\in\mathcal{U}}$ is a sequence of complex numbers satisfying $\|\{b^\lambda_{\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\|_{\ell^\infty}\le A_\lambda$. Then there exists a constant $C>0$ such that*

$$\big\|T_{\sigma^\lambda}(f_1,\cdots,f_m)\big\|_{L^{2/m}}\le CA_\lambda 2^{\lambda(m-1)n/2}\Big(\prod_{i\neq j,1\le i\le m}\|f_i\|_{L^2}\Big)\times\Big(\int_{\mathbb{R}^n}\big|\widehat{f}_j(\xi)\big|^2\sum_{\vec{k}\in\mathcal{U}}\big|\omega^\lambda_{k_j}(\xi)\big|^2 d\xi\Big)^{1/2}$$

for each $1\le j\le m$.

Remark 3. We should remark that $\sum_{\vec{k}\in\mathcal{U}}\big|\omega^\lambda_{k_j}(\xi)\big|^2$ could be very large since for each fixed k_j there may exist many $k\in\mathcal{U}$ such that $\mathcal{P}_j(k)=k_j$. This is how the support of $\mathcal{U}$ effects the norm of the multilinear operator T_{σ^λ}. A more exact estimate of this quantity relies on the structure of $\mathcal{U}$. See (20) and (21) below for some related calculations.

Proof. Without loss of generality, we may assume $j=1$. In view of (15), σ^λ can be written as

$$\sigma^\lambda(\vec{\xi})=\sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{U}}\omega^\lambda_{k_2}(\xi_2)\cdots\omega^\lambda_{k_m}(\xi_m)\sum_{k_1\in Col^{\mathcal{U}}_{\vec{k}^{*1}}}b^\lambda_{\vec{k}}\omega^\lambda_{k_1}(\xi_1),$$

and this yields that

$$T_{\sigma^\lambda}\big(f_1,\dots,f_m\big)(x) = \sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{U}} \Big(\prod_{i=2}^m L_{k_i}^\lambda f_i(x)\Big) \sum_{k_1\in Col^{\mathcal{U}}_{\vec{k}^{*1}}} b_{\vec{k}}^\lambda L_{k_1}^\lambda f_1(x),$$

where $L_k^\lambda f := (\omega_k^\lambda \widehat{f})^\vee$ for $k\in\mathbb{Z}^n$. Using the Cauchy-Schwarz inequality and Hölder's inequality, we obtain that

$$\begin{aligned}\big\|T_{\sigma^\lambda}(f_1,\dots,f_m)\big\|_{L^{2/m}} &\le \Big\|\Big(\sum_{\vec{k}^{*1}\in(\mathbb{Z}^n)^{m-1}} \Big|\prod_{i=2}^m L_{k_i}^\lambda f_i\Big|^2\Big)^{1/2}\Big\|_{L^{2/(m-1)}} \\ &\quad\times \Big\|\Big(\sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{U}} \Big|\sum_{k_1\in Col^{\mathcal{U}}_{\vec{k}^{*1}}} b_{\vec{k}}^\lambda L_{k_1}^\lambda f_1\Big|^2\Big)^{1/2}\Big\|_{L^2} \\ &=: I\times II.\end{aligned}$$

As a direct consequence of Plancherel's identity and (5), we have

$$\big\|\big\{L_k^\lambda f\big\}_{k\in\mathbb{Z}^n}\big\|_{L^2(\ell^2)} \lesssim 2^{\lambda n/2}\|f\|_{L^2}$$

and thus,

$$\begin{aligned} I &= \Big\|\prod_{i=2}^m\Big(\sum_{k_i\in\mathbb{Z}^n}|L_{k_i}^\lambda f_i|^2\Big)^{1/2}\Big\|_{L^{2/(m-1)}} \le \prod_{i=2}^m \big\|\big\{L_{k_i}^\lambda f_i\big\}_{k_i\in\mathbb{Z}^n}\big\|_{L^2(\ell^2)} \\ &\lesssim 2^{\lambda(m-1)n/2}\prod_{i=2}^m\|f_i\|_{L^2},\end{aligned}$$

where the first inequality is obtained by Hölder's inequality. Moreover, it follows from Plancherel's identity and the disjoint compact support property of $\{\omega_{k_1}^\lambda\}_{k_1\in\mathbb{Z}^n}$ that

$$\begin{aligned} II &\lesssim \Big(\sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{U}} \Big\|\widehat{f_1}\sum_{k_1\in Col^{\mathcal{U}}_{\vec{k}^{*1}}} b_{\vec{k}}^\lambda\omega_{k_1}^\lambda\Big\|_{L^2}^2\Big)^{1/2} \\ &\approx \Big(\int_{\mathbb{R}^n}|\widehat{f_1}(\xi)|^2\sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{U}}\sum_{k_1\in Col^{\mathcal{U}}_{\vec{k}^{*1}}} |b_{\vec{k}}^\lambda|^2|\omega_{k_1}^\lambda(\xi)|^2 d\xi\Big)^{1/2}\end{aligned}$$

and this is controlled by a constant multiple of

$$A_\lambda\Big(\int_{\mathbb{R}^n}|\widehat{f_1}(\xi)|^2\sum_{\vec{k}\in\mathcal{U}}|\omega_{k_1}^\lambda(\xi)|^2 d\xi\Big)^{1/2}$$

where (15) is applied. This completes the proof. □

3.1. **Proof of Proposition 2.1.** Let $N_1,\dots,N_{m-1}$ be positive numbers less than N, which will be chosen later. We separate $\mathcal{U}$ into m disjoint subsets

$$\begin{aligned}
\mathcal{U}^1 &:= \{\vec{k}\in\mathcal{U} : |Col^{\mathcal{U}}_{\vec{k}^{*1}}| > N_1\}\\
\mathcal{U}^2 &:= \{\vec{k}\in\mathcal{U}\setminus\mathcal{U}^1 : |Col^{\mathcal{U}}_{\vec{k}^{*1,2}}| > N_2\}\\
&\vdots\\
\mathcal{U}^{m-1} &:= \{\vec{k}\in\mathcal{U}\setminus(\mathcal{U}^1\cup\cdots\cup\mathcal{U}^{m-2}) : |Col^{\mathcal{U}}_{\vec{k}^{*1,\dots,m-1}}| > N_{m-1}\}\\
\mathcal{U}^m &:= \mathcal{U}\setminus(\mathcal{U}^1\cup\cdots\cup\mathcal{U}^{m-1})
\end{aligned}$$

and write

$$\sigma^\lambda = \sum_{j=1}^m \sum_{\vec{k}\in\mathcal{U}^j} b^\lambda_{\vec{k}}\,\omega^\lambda_{\vec{k}} =: \sum_{j=1}^m \sigma^\lambda_{(j)}.$$

Observe that for $1\le j\le m-1$, due to (15),

$$N \ge |\mathcal{U}^j| > N_j|\mathcal{P}_{*1,\dots,j}\mathcal{U}^j|,$$

which implies

$$|\mathcal{P}_{*1,\dots,j}\mathcal{U}^j| < NN_j^{-1}. \tag{17}$$

Moreover, for $2\le j\le m$ and $\vec{k}^{*1,\dots,j-1}\in\mathcal{P}_{*1,\dots,j-1}\mathcal{U}^j$

$$|Col^{\mathcal{U}^j}_{\vec{k}^{*1,\dots,j-1}}| \le N_{j-1}. \tag{18}$$

This is because if $\vec{k}^{*1,\dots,j-1} = (k_j,\dots,k_m)\in\mathcal{P}_{*1,\dots,j-1}\mathcal{U}^j$, then there exists $(l_1,\dots,l_{j-1})\in(\mathbb{Z}^n)^{j-1}$ such that $(l_1,\dots,l_{j-1},k_j,\dots,k_m)\in\mathcal{U}^j$ and thus $(l_1,\dots,l_{j-1},k_j,\dots,k_m)\notin\mathcal{U}^1\cup\cdots\cup\mathcal{U}^{j-1}$, which finally implies (18).

We now apply Lemma 3.1 to each $\sigma_{(j)}$, $1\le j\le m$, to obtain

$$\begin{aligned}
\big\|T_{\sigma^\lambda_{(j)}}(f_1,\cdots,f_m)\big\|_{L^{2/m}} &\le CA_\lambda 2^{\lambda(m-1)n/2}\prod_{i\ne j,1\le i\le m}\|f_i\|_{L^2}\\
&\quad\times\Big(\int_{\mathbb{R}^n}|\widehat{f_j}(\xi)|^2\sum_{\vec{k}\in\mathcal{U}^j}|\omega^\lambda_{k_j}(\xi)|^2 d\xi\Big)^{1/2}.
\end{aligned}\tag{19}$$

Note that

$$\sum_{\vec{k}\in\mathcal{U}^1}|\omega^\lambda_{k_1}(\xi)|^2 = \sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{U}^1}\Big(\sum_{k_1\in Col^{\mathcal{U}^1}_{\vec{k}^{*1}}}|\omega^\lambda_{k_1}(\xi)|^2\Big) \le 2^{\lambda n}|\mathcal{P}_{*1}\mathcal{U}^1| < 2^{\lambda n}NN_1^{-1} \tag{20}$$

where (15), (5), and (17) are applied. Similarly, when $2 \le j \le m-1$, we have

$$\sum_{\vec{k}\in\mathcal{U}^j} |\omega_{k_j}^{\lambda}(\xi)|^2 = \sum_{\vec{k}^{*1,\dots,j}\in\mathcal{P}_{*1,\dots,j}\mathcal{U}^j} \Big(\sum_{k_j\in\mathcal{P}_j Col^{\mathcal{U}^j}_{\vec{k}^{*1,\dots,j}}} |\omega_{k_j}^{\lambda}(\xi)|^2 |Col^{\mathcal{U}^j}_{\vec{k}^{*1,\dots,j-1}}| \Big)$$

$$\le 2^{\lambda n} N_{j-1} |\mathcal{P}_{*1,\dots,j}\mathcal{U}^j| \le 2^{\lambda n} N N_{j-1} N_j^{-1}, \tag{21}$$

using (16), (5), (18), and (17). For the last case $j = m$, it follows from (15), (5), and (18) that

$$\sum_{\vec{k}\in\mathcal{U}^m} |\omega_{k_m}^{\lambda}(\xi)|^2 = \sum_{k_m\in\mathcal{P}_m\mathcal{U}^m} |\omega_{k_m}^{\lambda}(\xi)|^2 |Col^{\mathcal{U}^m}_{k_m}| \le 2^{\lambda n} N_{m-1}.$$

Now we choose $N_1, \dots, N_{m-1}$ satisfying the identity

$$NN_1^{-1} = NN_1N_2^{-1} = NN_2N_3^{-1} = \cdots = NN_{m-2}N_{m-1}^{-1} = N_{m-1}. \tag{22}$$

Solving (22), we have

$$N_j = N^{j/m}, \qquad 1 \le j \le m-1$$

and this establishes

$$\sum_{\vec{k}\in\mathcal{U}^j} |\omega_{k_j}^{\lambda}(\xi)|^2 \le 2^{\lambda n} N^{(m-1)/m}, \qquad 1 \le j \le m,$$

which further implies

$$\Big(\int_{\mathbb{R}^n} |\widehat{f_j}(\xi)|^2 \sum_{\vec{k}\in\mathcal{U}^j} |\omega_{k_j}^{\lambda}(\xi)|^2 d\xi \Big)^{1/2} \le 2^{\lambda n/2} N^{(m-1)/2m} \|f_j\|_{L^2}.$$

Then this, together with (19), proves

$$\big\| T_{\sigma^{\lambda}_{(j)}}(f_1, \dots, f_m) \big\|_{L^{2/m}} \lesssim A_\lambda N^{(m-1)/2m} 2^{\lambda mn/2} \prod_{i=1}^{m} \|f_i\|_{L^2},$$

as desired.

3.2. **Proof of Proposition 2.4.** Let us recall some important notations first.

$$\mathcal{V}_l^{\mu} := \big\{ \vec{k} \in \mathcal{V}^{\mu} : |k_1|, \dots, |k_l| \ge M > |k_{l+1}|, \dots, |k_m| \big\},$$

where M is positive, and $L_k^{\lambda} f := \big(\omega_k^{\lambda} \widehat{f}\big)^{\vee}$.

We observe that

$$\big|\mathcal{P}_{*1,\dots,l}\mathcal{V}_l^{\mu}\big| \le M^{n(m-l)} \qquad \text{for } \mu \ge 0, \tag{23}$$

$$L_k^{\lambda,\gamma} f(x) = L_k^{\lambda}\big(f(\cdot/2^{\gamma})\big)(2^{\gamma}x),$$

and

$$|L_k^{\lambda,\gamma} f(x)| \lesssim 2^{\lambda n/2} \mathcal{M} f(x) \qquad \text{for } k \in \mathbb{Z}^n.$$

Here $\mathcal{M}$ is the Hardy-Littlewood maximal operator, defined by $\mathcal{M}f(x) := \sup_{Q\ni x}|Q|^{-1}\int_Q |f(y)|dy$, where the supremum is taken over all cubes containing x. This last inequality is possible as we can assume that ω is smooth enough. Then in view of (15) we can write

$$\Big|\sum_{\vec{k}\in\mathcal{V}_l^{\lambda+\mu}} b_{\vec{k}}^{\lambda,\gamma,\mu}\Big(\prod_{j=1}^{l} L_{k_j}^{\lambda,\gamma} f_j^{\lambda,\gamma,\mu}(x)\Big)\Big(\prod_{j=l+1}^{m} L_{k_j}^{\lambda,\gamma} f_j(x)\Big)\Big|$$

(24)

$$\lesssim 2^{\lambda(m-l)n/2}\Big[\prod_{j=l+1}^{m}\mathcal{M}f_j(x)\Big]$$
$$\times\sum_{\vec{k}^{*1,\dots,l}\in\mathcal{P}_{*1,\dots,l}\mathcal{V}_l^{\lambda+\mu}}\Big|\sum_{\vec{k}^{1,\dots,l}\in Col_{\vec{k}^{*1,\dots,l}}^{\mathcal{V}_l^{\lambda+\mu}}} b_{\vec{k}}^{\lambda,\gamma,\mu}\Big(\prod_{j=1}^{l} L_{k_j}^{\lambda}\big(f_j^{\lambda,\gamma,\mu}(\cdot/2^{\gamma})\big)(2^{\gamma}x)\Big)\Big|.$$

When $l=1$, using (24), Hölder's inequality, (Minkowski inequality for $r<2$), and the L^2 boundedness of $\mathcal{M}$, we obtain

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{k}\in\mathcal{V}_1^{\lambda+\mu}} b_{\vec{k}}^{\lambda,\gamma,\mu} L_{k_1}^{\lambda,\gamma} f_1^{\lambda,\gamma,\mu}\Big(\prod_{j=2}^{m} L_{k_j}^{\lambda,\gamma} f_j\Big)\Big|^r\Big)^{1/r}\Big\|_{L^{2/m}}$$
$$\lesssim 2^{\lambda(m-1)n/2}\Big(\prod_{j=2}^{m}\|f_j\|_{L^2}\Big)$$
$$\times\sum_{\vec{k}^{*1}\in\mathcal{P}_{*1}\mathcal{V}_1^{\lambda+\mu}}\Big(\sum_{\gamma\in\mathbb{Z}}\Big\|\sum_{k_1\in Col_{\vec{k}^{*1}}^{\mathcal{V}_1^{\lambda+\mu}}} b_{\vec{k}}^{\lambda,\gamma,\mu} L_{k_1}^{\lambda}\big(f_1^{\lambda,\gamma,\mu}(\cdot/2^{\gamma})\big)(2^{\gamma}\cdot)\Big\|_{L^2}^r\Big)^{1/r}.$$

A change of variables and Plancherel's identity yield that

$$\Big\|\sum_{k_1\in Col_{\vec{k}^{*1}}^{\mathcal{V}_1^{\lambda+\mu}}} b_{\vec{k}}^{\lambda,\gamma,\mu} L_{k_1}^{\lambda}\big(f_1^{\lambda,\gamma,\mu}(\cdot/2^{\gamma})\big)(2^{\gamma}\cdot)\Big\|_{L^2}\le A_{\lambda,\mu}2^{\lambda n/2}\|f_1^{\lambda,\gamma,\mu}\|_{L^2},$$

which combined with (23) proves the first estimate.

Similarly, for $0<q<\frac{2m}{m-1}$ and $2\le l\le m$, we can see

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\Big|\sum_{\vec{k}\in\mathcal{V}_l^{\lambda+\mu}} b_{\vec{k}}^{\lambda,\gamma,\mu}\Big(\prod_{j=1}^{l} L_{k_j}^{\lambda,\gamma} f_j^{\lambda,\gamma,\mu}\Big)\Big(\prod_{j=l+1}^{m} L_{k_j}^{\lambda,\gamma} f_j\Big)\Big|\Big\|_{L^{2/m}}$$
$$\lesssim 2^{\lambda(m-l)n/2}\Big(\prod_{j=l+1}^{m}\|f_j\|_{L^2}\Big)\quad\times$$

$$\sum_{\vec{k}^{*1,\dots,l} \in \mathcal{P}_{*1,\dots,l}\mathcal{V}_l^{\lambda+\mu}} \Big\| \sum_{\gamma\in\mathbb{Z}} \Big| \sum_{\vec{k}^{1,\dots,l} \in Col_{\vec{k}^{*1,\dots,l}}^{\mathcal{V}_l^{\lambda+\mu}}} b_{\vec{k}}^{\lambda,\gamma,\mu} \Big[\prod_{j=1}^{l} L_{k_i}^{\lambda}\big(f_j^{\lambda,\gamma,\mu}(\cdot/2^{\gamma})\big)(2^{\gamma}\cdot)\Big]\Big|\Big\|_{L^{2/l}}.$$

The $L^{2/l}$ norm is clearly dominated by

$$\Big(\sum_{\gamma\in\mathbb{Z}} \Big\| \sum_{\vec{k}^{1,\dots,l} \in Col_{\vec{k}^{*1,\dots,l}}^{\mathcal{V}_l^{\lambda+\mu}}} b_{\vec{k}}^{\lambda,\gamma,\mu} \Big(\prod_{j=1}^{l} L_{k_j}^{\lambda}\big(f_j^{\lambda,\gamma,\mu}(\cdot/2^{\gamma})\big)(2^{\gamma}\cdot)\Big)\Big\|_{L^{2/l}}^{2/l}\Big)^{l/2}$$

since $2/l \le 1$, and now we apply a change of variables, Proposition 2.2, and Hölder's inequality to obtain that the above expression is less than

$$\begin{aligned} &A_{\lambda,\mu}^{1-\frac{(l-1)q}{2l}} B_{\lambda,\mu,q}^{\frac{(l-1)q}{2l}} 2^{\lambda l n/2} \Big(\sum_{\gamma\in\mathbb{Z}} 2^{-\gamma n}\Big(\prod_{j=1}^{l} \|f_j^{\lambda,\gamma,\mu}(\cdot/2^{\gamma})\|_{L^2}^{2/l}\Big)\Big)^{l/2} \\ &\le A_{\lambda,\mu}^{1-\frac{(l-1)q}{2l}} B_{\lambda,\mu,q}^{\frac{(l-1)q}{2l}} 2^{\lambda l n/2} \prod_{j=1}^{l} \Big(\sum_{\gamma\in\mathbb{Z}} \|f_j^{\lambda,\gamma,\mu}\|_{L^2}^{2}\Big)^{1/2}. \end{aligned}$$

This completes the proof of the second statement.

The proof of the third one is essentially the same as the above argument except that we apply Proposition 2.3 instead of Proposition 2.2 in the last step.

4. Compactly supported wavelets

Typical functions possessing properties (i) and (ii) in Section 2 are the compactly supported wavelets constructed by Daubechies [8]; their construction is contained in the books of Meyer [30] and Daubechies [9]. Wavelets have been used to study singular integrals in different settings; see for instance [31], [15], [12], and [17]. For the purposes of this paper, we need smooth wavelets with compact supports but also of product type, like (6). The construction of such orthonormal bases is carefully presented in Triebel [36], but for the reader's sake we provide an outline. For any fixed $M \in \mathbb{N}$ there exist real compactly supported functions ψ_F, ψ_M in $\mathscr{C}^M(\mathbb{R})$ satisfying the following properties:

(a) $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$
(b) $\int_{\mathbb{R}} x^{\alpha}\psi_M(x)dx = 0$ for all $0 \le \alpha \le M$
(c) If $\Psi_{\vec{G}}$ is a function on $\mathbb{R}^{mn}$, defined by

$$\Psi_{\vec{G}}(\vec{x}) := \psi_{g_1}(x_1)\cdots\psi_{g_{mn}}(x_{mn})$$

for $\vec{x} := (x_1,\dots,x_{mn}) \in \mathbb{R}^{mn}$ and $\vec{G} := (g_1,\dots,g_{mn})$ in the set

$$\mathcal{I} := \big\{\vec{G} := (g_1,\dots,g_{mn}) : g_i \in \{F,M\}\big\},$$

then the family of functions

$$\bigcup_{\lambda\in\mathbb{N}_0}\bigcup_{\vec{k}\in\mathbb{Z}^{mn}}\big\{2^{\lambda mn/2}\Psi_{\vec{G}}(2^{\lambda}\vec{x}-\vec{k}):\vec{G}\in\mathcal{I}^{\lambda}\big\}$$

forms an orthonormal basis of $L^2(\mathbb{R}^{mn})$, where $\mathcal{I}^0:=\mathcal{I}$ and $\mathcal{I}^{\lambda}:=\mathcal{I}\setminus\{(F,\dots,F)\}$ for $\lambda\ge 1$.

Fix $1<q<\infty$ and $s\ge 0$. Let $\|F\|_{L^q_s(\mathbb{R}^{mn})}$ denote the Sobolev space norm defined as the $L^q((\mathbb{R}^n)^m)$ norm of $(\vec{I}-\vec{\Delta})^{s/2}F$, where $\vec{\Delta}$ is the Laplacian of a function F on $(\mathbb{R}^n)^m$. It is also shown in [35] that if M is sufficiently large and F is a tempered distribution on $\mathbb{R}^{mn}$ lying in $L^q_s(\mathbb{R}^{mn})$, then F can be represented as

$$F(\vec{x})=\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^{\lambda}}\sum_{\vec{k}\in\mathbb{Z}^{mn}}b^{\lambda}_{\vec{G},\vec{k}}2^{\lambda mn/2}\Psi_{\vec{G}}(2^{\lambda}\vec{x}-\vec{k}) \tag{25}$$

and

$$\Big\|\Big(\sum_{\vec{G},\,\vec{k}}\big|b^{\lambda}_{\vec{G},\vec{k}}\Psi^{\lambda}_{\vec{G},\vec{k}}\big|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^{mn})}\le C2^{-s\lambda}\|F\|_{L^q_s(\mathbb{R}^{mn})},$$

where $\Psi^{\lambda}_{\vec{G},\vec{k}}(\vec{x})=2^{\lambda mn/2}\Psi_{\vec{G}}(2^{\lambda}\vec{x}-\vec{k})$, and

$$b^{\lambda}_{\vec{G},\vec{k}}:=\int_{\mathbb{R}^{mn}}F(\vec{x})\Psi^{\lambda}_{\vec{G},\vec{k}}(\vec{x})d\vec{x}.$$

Moreover, it follows from the last estimate and the disjoint support property of the $\Psi^{\lambda}_{\vec{G},\vec{k}}$'s that

$$\begin{aligned}\big\|\{b^{\lambda}_{\vec{G},\vec{k}}\}_{\vec{k}\in\mathbb{Z}^{mn}}\big\|_{\ell^q}&\approx\Big(2^{\lambda mn(1-q/2)}\int_{\mathbb{R}^{mn}}\Big(\sum_{\vec{k}}\big|b^{\lambda}_{\vec{G},\vec{k}}\Psi^{\lambda}_{\vec{G},\vec{k}}(\vec{x})\big|^2\Big)^{q/2}d\vec{x}\Big)^{1/q}\\&\lesssim 2^{-\lambda(s-mn/q+mn/2)}\|F\|_{L^q_s(\mathbb{R}^{mn})}.\end{aligned} \tag{26}$$

We will write $\vec{G}:=(G_1,\dots,G_m)\in(\{F,M\}^n)^m$ and

$$\Psi_{\vec{G}}(\vec{\xi})=\Psi_{G_1}(\xi_1)\cdots\Psi_{G_m}(\xi_m).$$

For each $\vec{k}:=(k_1,\dots,k_m)\in(\mathbb{Z}^n)^m$ and $\lambda\in\mathbb{N}_0$, let

$$\Psi^{\lambda}_{G_i,k_i}(\xi_i):=2^{\lambda n/2}\Psi_{G_i}(2^{\lambda}\xi_i-k_i),\qquad 1\le i\le m$$

and

$$\Psi^{\lambda}_{\vec{G},\vec{k}}(\vec{\xi}):=\Psi^{\lambda}_{G_1,k_1}(\xi_1)\cdots\Psi^{\lambda}_{G_m,k_m}(\xi_m).$$

We also assume that the support of ψ_{g_i} is contained in $\{\xi\in\mathbb{R}:|\xi|\le C_0\}$ for some $C_0>1$, which implies that

$$\mathrm{Supp}(\Psi^{\lambda}_{G_i,k_i})\subset\big\{\xi_i\in\mathbb{R}^n:|2^{\lambda}\xi_i-k_i|\le C_0\sqrt{n}\big\}. \tag{27}$$

In other words, the support of $\Psi^\lambda_{G_i,k_i}$ is contained in the ball centered at $2^{-\lambda}k_i$ and radius $C_0\sqrt{n}2^{-\lambda}$.

5. Proof of Theorem 1.1

Using (25) with $s=0$, we decompose σ as

$$\sigma(\vec{\xi}) = \sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sum_{\vec{k}\in(\mathbb{Z}^n)^m} b^\lambda_{\vec{G},\vec{k}}\Psi^\lambda_{G_1,k_1}(\xi_1)\cdots\Psi^\lambda_{G_m,k_m}(\xi_m) =: \sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sigma^\lambda_{\vec{G}}(\vec{\xi})$$

where $b^\lambda_{\vec{G},\vec{k}} := \int_{(\mathbb{R}^n)^m}\sigma(\vec{\xi})\Psi^\lambda_{\vec{G},\vec{k}}(\vec{\xi})d\vec{\xi}$. As an immediate consequence of Proposition 2.2, we have

$$\big\|T_{\sigma^\lambda_{\vec{G}}}(f_1,\dots,f_m)\big\|_{L^{2/m}} \lesssim \big\|\{b^\lambda_{\vec{G},\vec{k}}\}_{\vec{k}}\big\|_{\ell^\infty}^{1-\frac{(m-1)q}{2m}}\big\|\{b^\lambda_{\vec{G},\vec{k}}\}_{\vec{k}}\big\|_{\ell^q}^{\frac{(m-1)q}{2m}}2^{\lambda mn/2}\prod_{j=1}^m\|f_j\|_{L^2}.$$

We first observe that (26) yields that

$$\big\|\{b^\lambda_{\vec{G},\vec{k}}\}_{\vec{k}}\big\|_{\ell^q} \lesssim 2^{\lambda mn(1/q-1/2)}\|\sigma\|_{L^q((\mathbb{R}^n)^m)}.$$

In addition, as $\sigma\in\mathscr{C}^{M_q}((\mathbb{R}^n)^m)$, using this property, the M_q vanishing moment condition of $\Psi^\lambda_{\vec{G},\vec{k}}$ in conjunction with Taylor's formula, an argument similar to [19, Lemma 2.1] and [17, Lemma 7] yields

$$\big\|\{b^\lambda_{\vec{G},\vec{k}}\}_{\vec{k}}\big\|_{\ell^\infty} \lesssim 2^{-\lambda(M_q+mn/2)}D_0.$$

Here we choose the number of vanishing moment as M_q so that we can obtain sufficient decay, which will be useful in summing over λ later. Therefore, we finally arrive at the estimate

$$\begin{aligned}&\big\|T_{\sigma^\lambda_{\vec{G}}}(f_1,\dots,f_m)\big\|_{L^{2/m}}\\ &\lesssim 2^{-\lambda\left(M_q(1-\frac{(m-1)q}{2m})-\frac{n(m-1)}{2}\right)}D_0^{1-\frac{(m-1)q}{2m}}\|\sigma\|_{L^q((\mathbb{R}^n)^m)}^{\frac{(m-1)q}{2m}}\prod_{j=1}^m\|f_j\|_{L^2},\end{aligned}$$

which in turn implies that

$$\begin{aligned}&\big\|T_\sigma(f_1,\dots,f_m)\big\|_{L^{2/m}}\\ &\le\Big(\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\big\|T_{\sigma^\lambda_{\vec{G}}}(f_1,\dots,f_m)\big\|_{L^{2/m}}^{2/m}\Big)^{m/2}\\ &\lesssim D_0^{1-\frac{(m-1)q}{2m}}\|\sigma\|_{L^q((\mathbb{R}^n)^m)}^{\frac{(m-1)q}{2m}}\Big(\sum_{\lambda\in\mathbb{N}_0}2^{-\lambda(M_q(1-\frac{(m-1)q}{2m})-\frac{n(m-1)}{2})\frac{2}{m}}\Big)^{m/2}\prod_{j=1}^m\|f_j\|_{L^2}.\end{aligned}$$

Since $M_q > \frac{m(m-1)n}{2m-(m-1)q}$, the sum over λ converges and completes the proof.

6. Proof of Theorem 1.2

Without loss of generality, we may assume $\frac{2m}{m+1} < q < 2$ as $L^r(\mathbb{S}^{mn-1}) \subset L^q(\mathbb{S}^{mn-1})$ for $r \geq q$. We first utilize a dyadic decomposition introduced by Duoandikoetxea and Rubio de Francia [14]. Recall that $\Phi^{(m)}$ is a Schwartz function such that $\widehat{\Phi^{(m)}}$ is supported in the annulus $\{\vec{\xi} \in (\mathbb{R}^n)^m : 1/2 \leq |\vec{\xi}| \leq 2\}$ and $\sum_{j\in\mathbb{Z}} \widehat{\Phi_j^{(m)}}(\vec{\xi}) = 1$ for $\vec{\xi} \neq \vec{0}$ where $\widehat{\Phi_j^{(m)}}(\vec{\xi}) := \widehat{\Phi^{(m)}}(\vec{\xi}/2^j)$.

For $\gamma \in \mathbb{Z}$ let

$$K^\gamma(\vec{y}) := \widehat{\Phi^{(m)}}(2^\gamma \vec{y})K(\vec{y}), \quad \vec{y} \in (\mathbb{R}^n)^m$$

and then we observe that $K^\gamma(\vec{y}) = 2^{\gamma mn}K^0(2^\gamma \vec{y})$. For $\mu \in \mathbb{Z}$ we define

$$K_\mu^\gamma(y) := \Phi_{\mu+\gamma}^{(m)} * K^\gamma(y) = 2^{\gamma mn}[\Phi_\mu^{(m)} * K^0](2^\gamma y).$$

It follows from this definition that

$$\widehat{K_\mu^\gamma}(\vec{\xi}) = \widehat{\Phi^{(m)}}(2^{-(\mu+\gamma)}\vec{\xi})\widehat{K^0}(2^{-\gamma}\vec{\xi}) = \widehat{K_\mu^0}(2^{-\gamma}\vec{\xi}),$$

which implies that $\widehat{K_\mu^\gamma}$ is bounded uniformly in γ while they have almost disjoint supports, so it is natural to add them together as follows,

$$K_\mu(\vec{y}) := \sum_{\gamma\in\mathbb{Z}} K_\mu^\gamma(\vec{y}).$$

We define

$$\mathcal{L}_\mu\big(f_1,\dots,f_m\big)(x) := \int_{(\mathbb{R}^n)^m} K_\mu(\vec{y}) \prod_{j=1}^m f_j(x-y_j)\, d\vec{y}, \quad x \in \mathbb{R}^n$$

and write

$$\begin{aligned} \big\|\mathcal{L}_\Omega(f_1,\dots,f_m)\big\|_{L^{2/m}} \lesssim& \Big\| \sum_{\mu\in\mathbb{Z}:2^{\mu-10}\leq C_0\sqrt{mn}} \mathcal{L}_\mu(f_1,\dots,f_m)\Big\|_{L^{2/m}} \\ &+ \Big\| \sum_{\mu\in\mathbb{Z}:2^{\mu-10}> C_0\sqrt{mn}} \mathcal{L}_\mu(f_1,\dots,f_m)\Big\|_{L^{2/m}} \end{aligned} \tag{28}$$

where C_0 is the constant that appeared in (27).

The analysis of $\mathcal{L}_\Omega$ will be reduced to analyzing $\mathcal{L}_\mu$ from the frequency side. More precisely, we will need the well-known result

$$|\partial^\alpha \widehat{K^0}(\vec{\xi}\,)| \leq C_\alpha \|\Omega\|_{L^q} \begin{cases} \min(|\vec{\xi}\,|, |\vec{\xi}\,|^{-\delta}) & \alpha = 0 \\ \min(1, |\vec{\xi}\,|^{-\delta}) & \alpha \neq 0. \end{cases} \tag{29}$$

See [13, Lemma 8.20] for detailed calculations. With these estimates in hand, using the Cofiman-Meyer theorem [4] and the argument in [17, Proposition 3], we can prove that

(30)
$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^p}\lesssim\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\Big(\prod_{j=1}^m\|f_j\|_{L^{p_j}}\Big)\begin{cases}2^{(mn-\delta)\mu}, & \mu\geq 0\\ 2^{(1-\delta)\mu}, & \mu<0\end{cases}$$

for $0<\delta<1/q'$. This implies that

$$\Big\|\sum_{\mu\in\mathbb{Z}:2^{\mu-10}\leq C_0\sqrt{mn}}\mathcal{L}_\mu(f_1,\dots,f_m)\Big\|_{L^{2/m}}\lesssim\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^2}.$$

It remains to bound the term (28), but this can be reduced to proving that for $2^{\mu-10}>C_0\sqrt{mn}$, there exists $\varepsilon_0>0$ such that

$$\big\|\mathcal{L}_\mu(f_1,\dots,f_m)\big\|_{L^{2/m}}\lesssim 2^{-\varepsilon_0\mu}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^2},\tag{31}$$

which compensate the estimate (30) for $\mu\geq0$. Recall that

$$\widehat{K_\mu}(\vec{\xi})=\sum_{\gamma\in\mathbb{Z}}\widehat{K_\mu^0}(\vec{\xi}/2^\gamma)$$

and

$$\mathrm{Supp}\widehat{K_\mu^0}\subset\big\{\vec{\xi}\in(\mathbb{R}^n)^m:2^{\mu-1}\leq|\vec{\xi}|\leq2^{\mu+1}\big\}.\tag{32}$$

Using (25), $\widehat{K_\mu^0}$ can be written as

$$\widehat{K_\mu^0}(\vec{\xi})=\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sum_{\vec{k}\in(\mathbb{Z}^n)^m}b_{\vec{G},\vec{k}}^{\lambda,\mu}\Psi_{G_1,k_1}^\lambda(\xi_1)\cdots\Psi_{G_m,k_m}^\lambda(\xi_m)\tag{33}$$

where

$$b_{\vec{G},\vec{k}}^{\lambda,\mu}:=\int_{(\mathbb{R}^n)^m}\widehat{K_\mu^0}(\vec{\xi})\Psi_{\vec{G},\vec{k}}^\lambda(\vec{\xi})d\vec{\xi}.$$

By the vanishing moments of the mother wavelet ψ_M and (29) we have

$$\big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^\infty}\lesssim2^{-\delta\mu}2^{-\lambda(M+1+mn)}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\tag{34}$$

where M is the number of vanishing moments of $\Psi_{\vec{G}}$ and $0<\delta<1/q'$; see [17, Lemma 7] for the related calculation. In addition, (26), the Hausdorff-Young inequality, and Young's inequality prove that

$$\begin{aligned}\big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^{q'}}&\lesssim2^{-\lambda mn(1/2-1/q')}\|\widehat{K_\mu^0}\|_{L^{q'}}\\&\lesssim2^{-\lambda mn(1/q-1/2)}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}.\end{aligned}\tag{35}$$

Furthermore, if $2^{\mu-10} > C_0\sqrt{mn}$, then we may replace $\vec{k} \in (\mathbb{Z}^n)^m$ in (33) by $2^{\lambda+\mu-2} \le |\vec{k}| \le 2^{\lambda+\mu+2}$ due to (32) and the compact support condition of $\Psi^{\lambda}_{\vec{G},\vec{k}}$. Therefore the proof of (31) can be reduced to the inequality

$$\begin{aligned} &\Big\| \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sum_{\gamma\in\mathbb{Z}} \sum_{\vec{k}\in\mathcal{U}^{\lambda+\mu}} b^{\lambda,\mu}_{\vec{G},\vec{k}} \prod_{j=1}^m L^{\lambda,\gamma}_{G_j,k_j} f_j \Big\|_{L^{2/m}} \\ &\lesssim 2^{-\varepsilon_0\mu} \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} \prod_{j=1}^m \|f_j\|_{L^2} \end{aligned} \tag{36}$$

where

$$\mathcal{U}^{\lambda+\mu} := \big\{ \vec{k} \in (\mathbb{Z}^n)^m : 2^{\lambda+\mu-2} \le |\vec{k}| \le 2^{\lambda+\mu+2},\ |k_1| \ge \cdots \ge |k_m| \big\}$$

and

$$L^{\lambda,\gamma}_{G,k} f := \big(\Psi^{\lambda}_{G,k}(\cdot/2^\gamma) \widehat{f} \big)^{\vee}. \tag{37}$$

Here, it is additionally assumed that $|k_1| \ge \cdots \ge |k_m|$ in $\mathcal{U}^{\lambda+\mu}$ as the remaining cases follow by symmetry and there are at most $m!$ many such cases. Then we note that $\mathcal{U}^{\lambda+\mu}$ can be expressed as the union of m disjoint subsets

$$\begin{aligned} \mathcal{U}^{\lambda+\mu}_1 &:= \{ \vec{k} \in \mathcal{U}^{\lambda+\mu} : |k_1| \ge 2C_0\sqrt{n} > |k_2| \ge \cdots \ge |k_m| \} \\ \mathcal{U}^{\lambda+\mu}_2 &:= \{ \vec{k} \in \mathcal{U}^{\lambda+\mu} : |k_1| \ge |k_2| \ge 2C_0\sqrt{n} > |k_3| \ge \cdots \ge |k_m| \} \\ &\vdots \\ \mathcal{U}^{\lambda+\mu}_m &:= \{ \vec{k} \in \mathcal{U}^{\lambda+\mu} : |k_1| \ge \cdots \ge |k_m| \ge 2C_0\sqrt{n} \}. \end{aligned}$$

We remark that the case $|k_1| < 2C_0\sqrt{n}$ is excluded because it implies $|\vec{k}| < 2C_0\sqrt{mn}$ for which $\vec{k}$ is not contained in $\mathcal{U}^{\lambda+\mu}$ as $2^{\mu-10} > C_0\sqrt{mn}$.

The function in the left-hand side of (36) could be written as

$$\sum_{l=1}^m \sum_{\lambda\in\mathbb{N}_0} \sum_{\vec{G}\in\mathcal{I}^\lambda} \sum_{\gamma\in\mathbb{Z}} \mathcal{T}^{\lambda,\gamma,\mu}_{\vec{G},l}(f_1,\dots,f_m)$$

where

$$\mathcal{T}^{\lambda,\gamma,\mu}_{\vec{G},l}(f_1,\dots,f_m) := \sum_{\vec{k}\in\mathcal{U}^{\lambda+\mu}_l} b^{\lambda,\mu}_{\vec{G},\vec{k}} \Big(\prod_{j=1}^m L^{\lambda,\gamma}_{G_j,k_j} f_j \Big). \tag{38}$$

Observe that when $\vec{k} \in \mathcal{U}^{\lambda+\mu}_l$,

$$L^{\lambda,\gamma}_{G_j,k_j} f_j = L^{\lambda,\gamma}_{G_j,k_j} f_j^{\lambda,\gamma,\mu} \qquad \text{for } 1 \le j \le l \tag{39}$$

due to the support of $\Psi^{\lambda}_{G_j,k_j}$, where $\widehat{f_j^{\lambda,\gamma,\mu}}(\xi_j) := \widehat{f_j}(\xi_j)\chi_{C_0\sqrt{n}2^{\gamma-\lambda}\le|\xi_j|\le 2^{\gamma+\mu+3}}$. Moreover, it is easy to show that for $\mu\ge 10$ and $\lambda\in\mathbb{N}_0$,

$$\Big(\sum_{\gamma\in\mathbb{Z}}\|f_j^{\lambda,\gamma,\mu}\|_{L^2}^2\Big)^{1/2}\lesssim(\mu+\lambda)^{1/2}\|f_j\|_{L^2}\lesssim\mu^{1/2}(\lambda+1)^{1/2}\|f_j\|_{L^2} \tag{40}$$

where Plancherel's identity is applied in the first inequality and the factor $(\mu+\lambda)^{1/2}$ is due to the fact that each ξ is contained in $C(\mu+\lambda)$ annuli of the form $\{\xi_j\in\mathbb{R}^n:\ C_0\sqrt{n}2^{\gamma-\lambda}\le|\xi_j|\le 2^{\gamma+\mu+3}\}$.

Now we claim that for each $1\le l\le m$ there exists $\varepsilon_0, M_0>0$ such that

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\mathcal{T}^{\lambda,\gamma,\mu}_{\vec{G},l}\big(f_1,\dots,f_m\big)\Big\|_{L^{2/m}}\lesssim\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}2^{-\varepsilon_0\mu}2^{-\lambda M_0}\prod_{j=1}^m\|f_j\|_{L^2}. \tag{41}$$

Then the left-hand side of (36) is controlled by a constant times

$$\begin{aligned}&\Big(\sum_{l=1}^m\sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\Big\|\sum_{\gamma\in\mathbb{Z}}\mathcal{T}^{\lambda,\gamma,\mu}_{\vec{G},l}\big(f_1,\dots,f_m\big)\Big\|_{L^{2/m}}^{2/m}\Big)^{m/2}\\&\lesssim 2^{-\varepsilon_0\mu}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\prod_{j=1}^m\|f_j\|_{L^2},\end{aligned}$$

which completes the proof of (36). Therefore, it remains to prove (41).

6.1. **The case $l=1$.** We observe first that Proposition 2.4 implies

$$\begin{aligned}&\Big\|\sum_{\gamma\in\mathbb{Z}}\mathcal{T}^{\lambda,\gamma,\mu}_{\vec{G},1}\big(f_1,\dots,f_m\big)\Big\|_{L^{2/m}}\le\Big\|\sum_{\gamma\in\mathbb{Z}}\big|\mathcal{T}^{\lambda,\gamma,\mu}_{\vec{G},1}\big(f_1,\dots,f_m\big)\big|\Big\|_{L^{2/m}}\\&\lesssim\big\|\{b^{\lambda,\mu}_{\vec{G},\vec{k}}\}_{\vec{k}}\big\|_{\ell^\infty}2^{\lambda mn/2}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_1^{\gamma,\lambda,\mu}\|_{L^2}\Big)\prod_{j=1}^m\|f_j\|_{L^2}\\&\lesssim 2^{-\delta\mu}2^{-\lambda(M+1+mn/2)}\|\Omega\|_{L^q(\mathbb{S}^{mn-1})}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_1^{\lambda,\gamma,\mu}\|_{L^2}\Big)\prod_{j=1}^m\|f_j\|_{L^2}.\end{aligned}$$

It is unlikely that we can bound $\sum_{\gamma\in\mathbb{Z}}\|f_1^{\lambda,\gamma,\mu}\|_{L^2}$ by $\|f_1\|_{L^2}$ easily. On the other hand, since f_1 is a Schwartz function, we have

$$\begin{aligned}\Big\|f_1^{\lambda,\gamma,\mu}\Big\|_{L^2}&=\Big\|\widehat{f_1^{\lambda,\gamma,\mu}}\Big\|_{L^2}=\Big(\int_{C_0\sqrt{n}2^{\gamma-\lambda}\le|\xi|\le 2^{\gamma+\mu+3}}|\widehat{f_1}(\xi)|^2d\xi\Big)^{1/2}\\&\lesssim_N\begin{cases}2^{(\gamma+\mu)n/2}, & \gamma<0\\ 2^{-(\gamma-\lambda)(N-n/2)}, & \gamma\ge 0\end{cases}\end{aligned} \tag{42}$$

for sufficiently large $N>n/2$, which yields that

$$\sum_{\gamma\in\mathbb{Z}}\|f_1^{\lambda,\gamma,\mu}\|_{L^2}$$

is finite (of course, this depends on λ, μ, and f_1). Therefore, we also have

$$\sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big) \in L^{2/m} \tag{43}$$

To improve the previous argument, we will use the square function characterization of Hardy spaces, which relies on the fact that if $\widehat{g_\gamma}$ is supported on $\{\xi\in\mathbb{R}^n : C^{-1}2^{\gamma+\mu}\le|\xi|\le C2^{\gamma+\mu}\}$ for some $C>1$ and $\mu\in\mathbb{Z}$, then

$$\Big\|\Big\{\Phi_j^{(1)}*\Big(\sum_{\gamma\in\mathbb{Z}} g_\gamma\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^q)} \lesssim_C \big\|\{g_j\}_{j\in\mathbb{Z}}\big\|_{L^p(\ell^q)} \quad \text{uniformly in } \mu \tag{44}$$

for $0<p<\infty$. The proof of (44) is elementary and standard, so we just provide a sketch of that. Due to the Fourier support conditions of both $\Phi_j^{(1)}$ and g_γ, the left-hand side of (44) would be

$$\Big\|\Big\{\Phi_j^{(1)}*\Big(\sum_{\gamma=j-\mu-C'}^{j-\mu+C'} g_\gamma\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^p(\ell^q)}$$

for some constant $C'>1$. Then we use the estimate that for any $r>0$ and $j-\mu-C'\le\gamma\le j-\mu+C'$,

$$\big|\Phi_j^{(1)}*g_\gamma(x)\big| \lesssim_{r,C'} 2^{j(n/r-n)}\Big(\int_{\mathbb{R}^n}\big|\Phi_j^{(1)}(x-y)\big|^r\big|g_\gamma(y)\big|^r\,dy\Big)^{1/r} \lesssim \mathcal{M}_r g_\gamma(x)$$

uniformly in μ, where Bernstein's inequality is applied in the first estimate. Here, $\mathcal{M}_r g_\gamma := \big(\mathcal{M}(|g_\gamma|^r)\big)^{1/r}$ and $\mathcal{M}$ is the Littlewood-Paley maximal operator as before. Then (44) follows from choosing $0<r<p,q$ and applying Fefferman-Stein's maximal inequality for $\mathcal{M}_r$. See [17, (13)] and [37, Theorem 3.6] for a related argument.

Note that

$$2^{\lambda+\mu-3}\le 2^{\lambda+\mu-2}-2C_0\sqrt{mn}\le|\vec{k}|-(|k_2|^2+\cdots+|k_m|^2)^{1/2}\le|k_1|\le 2^{\lambda+\mu+2}$$

and this implies that

$$\mathrm{Supp}\big(\Psi_{G_1,k_1}^{\lambda}(\cdot/2^\gamma)\big)\subset\{\xi\in\mathbb{R}^n : 2^{\gamma+\mu-4}\le|\xi|\le 2^{\gamma+\mu+3}\}.$$

Moreover, since $|k_j|\le 2C_0\sqrt{n}$ for $2\le j\le m$ and $2^{\mu-10}>C_0\sqrt{mn}$,

$$\mathrm{Supp}\big(\Psi_{G_j,k_j}^{\lambda}(\cdot/2^\gamma)\big)\subset\{\xi\in\mathbb{R}^n : |\xi|\le m^{-1/2}2^{\gamma+\mu-8}\}.$$

Therefore, the Fourier transform of $\mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big)$ is supported in the set $\{\xi\in\mathbb{R}^n : 2^{\gamma+\mu-5}\le|\xi|\le 2^{\gamma+\mu+4}\}$ due to the definitions (37) and (38).

Using the Littlewood-Paley theory for Hardy spaces [16, Theorem 2.2.9], there exists a unique polynomial $Q^{\lambda,\mu,\vec{G}}(x)$ such that

$$\begin{aligned}
&\Big\| \sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big)-Q^{\lambda,\mu,\vec{G}}\Big\|_{L^{2/m}} \\
&\lesssim \Big\| \Big\{ \Phi_j^{(1)} * \Big(\sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big)\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^{2/m}(\ell^2)}
\end{aligned} \tag{45}$$

and then (44) and (39) yield that the above $L^{2/m}(\ell^2)$-norm is dominated by a constant multiple of

$$\Big\| \Big(\sum_{\gamma\in\mathbb{Z}} \big| \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1^{\lambda,\gamma,\mu},f_2,\dots,f_m\big)\big|^2\Big)^{1/2}\Big\|_{L^{2/m}}.$$

We now apply Proposition 2.4, (34), and (40) to bound the $L^{2/m}$-norm by

$$\begin{aligned}
&\big\| \{ b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^\infty} 2^{\lambda mn/2} \Big(\sum_{\gamma\in\mathbb{Z}} \big\| f_1^{\lambda,\gamma,\mu}\big\|_{L^2}^2\Big)^{1/2} \prod_{j=2}^m \|f_j\|_{L^2} \\
&\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} 2^{-\delta\mu}\mu^{1/2} 2^{-\lambda(M+1+mn/2)}(\lambda+1)^{1/2} \prod_{j=1}^m \|f_j\|_{L^2}.
\end{aligned}$$

This implies that the left-hand side of (45) is bounded by

$$\|\Omega\|_{L^q(\mathbb{S}^{mn-1})} 2^{-\epsilon_0\mu} 2^{-\lambda M_0} \prod_{j=1}^m \|f_j\|_{L^2}$$

for some $0<\epsilon_0<\delta$ and $0<M_0<M+1+mn/2$, and thus

$$\sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big)-Q^{\lambda,\mu,\vec{G}} \in L^{2/m}. \tag{46}$$

Recalling that $\sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big)\in L^{2/m}$, the polynomial $Q^{\lambda,\mu,\vec{G}}$ in (46) should be zero. In conclusion,

$$\begin{aligned}
&\Big\| \sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},1}^{\lambda,\gamma,\mu}\big(f_1,\dots,f_m\big)\Big\|_{L^{2/m}} \\
&\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} 2^{-\epsilon_0\mu} 2^{-\lambda M_0} \prod_{j=1}^m \|f_j\|_{L^2}.
\end{aligned}$$

This proves (41) for $l=1$.

6.2. **The case $2 \le l \le m$.** We apply (39), Proposition 2.4 with $2 < q' < \frac{2m}{m-1}$, (40), (34), and (35) to obtain that

$$\Big\| \sum_{\gamma\in\mathbb{Z}} \mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\Big\|_{L^{2/m}} \le \Big\| \sum_{\gamma\in\mathbb{Z}} \big|\mathcal{T}_{\vec{G},l}^{\lambda,\gamma,\mu}(f_1,\dots,f_m)\big|\Big\|_{L^{2/m}}$$

$$\lesssim \big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^\infty}^{1-\frac{(m-1)q'}{2m}} \big\|\{b_{\vec{G},\vec{k}}^{\lambda,\mu}\}_{\vec{k}}\big\|_{\ell^{q'}}^{\frac{(m-1)q'}{2m}} 2^{\lambda mn/2}(\lambda+1)^{l/2}\mu^{l/2}\prod_{j=1}^m \|f_j\|_{L^2}$$

(47)

$$\lesssim \|\Omega\|_{L^q(\mathbb{S}^{mn-1})} 2^{-\delta\mu(1-\frac{(m-1)q'}{2m})}\mu^{m/2}2^{-\lambda C_{M,m,n,q}}(\lambda+1)^{m/2}\prod_{j=1}^m \|f_j\|_{L^2}$$

where

$$C_{M,m,n,q} := (M+1+mn)\Big(1-\frac{(m-1)q'}{2m}\Big)+mn(1/q-1/2)\frac{(m-1)q'}{2m}-\frac{mn}{2}.$$

Here we used the embedding $\ell^{q'} \hookrightarrow \ell^\infty$ and the fact that $\frac{l-1}{2l} \le \frac{m-1}{2m}$. Then (41) follows from choosing M sufficiently large so that $C_{M,m,n,q} > 0$ since $1-\frac{(m-1)q'}{2m} > 0$.

7. Proof of Theorem 1.3

The strategy in this section is similar to that used in handling multilinear rough singular integrals in Section 6, but the decomposition is more delicate. We describe the decomposition first. Write

$$\sigma(\vec{\xi}) = \sum_{\gamma\in\mathbb{Z}} \sigma_\gamma(\vec{\xi}/2^\gamma)$$

where $\sigma_\gamma(\vec{\xi}) := \sigma(2^\gamma\vec{\xi})\widehat{\Phi^{(m)}}(\vec{\xi})$. Clearly,

$$\text{(48)} \qquad \operatorname{Supp}(\sigma_\gamma) \subset \{\vec{\xi} \in (\mathbb{Z}^n)^m : 1/2 \le |\vec{\xi}| \le 2\}$$

and according to (25),

$$\text{(49)} \qquad \sigma_\gamma(\vec{\xi}) = \sum_{\lambda\in\mathbb{N}_0}\sum_{\vec{G}\in\mathcal{I}^\lambda}\sum_{\vec{k}\in(\mathbb{Z}^n)^m} b_{\vec{G},\vec{k}}^{\lambda,\gamma}\Psi_{G_1,k_1}^\lambda(\xi_1)\cdots\Psi_{G_m,k_m}^\lambda(\xi_m)$$

where $b_{\vec{G},\vec{k}}^{\lambda,\gamma} := \int_{(\mathbb{R}^n)^m}\sigma_\gamma(\vec{\xi})\Psi_{\vec{G},\vec{k}}^\lambda(\vec{\xi})d\vec{\xi}$. Moreover, it follows from (26) that for $1 < q < \infty$ and $s \ge 0$

$$\text{(50)} \qquad \big\|\{b_{\vec{G},\vec{k}}^{\lambda,\gamma}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^q} \lesssim 2^{-\lambda(s-mn/q+mn/2)}\big\|\sigma(2^\gamma\vec{\cdot})\widehat{\Phi^{(m)}}\big\|_{L^q_s((\mathbb{R}^n)^m)}.$$

As we did in the proof of Theorem 1.2, it is enough to consider only the case $|k_1| \ge \cdots \ge |k_m|$. Therefore, we replace $\vec{k} \in (\mathbb{Z}^n)^m$ in (49) by $\vec{k} \in \mathcal{U} :=$

$\{\vec{k} \in (\mathbb{Z}^n)^m : |k_1| \geq \cdots \geq |k_m|\}$ and write

$$\begin{aligned}\sigma_\gamma(\vec{\xi}) = &\sum_{\lambda \in \mathbb{N}_0 : 2^\lambda \geq 2^8 C_0 m\sqrt{n}} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\vec{k} \in \mathcal{U}} b^{\lambda,\gamma}_{\vec{G},\vec{k}} \Psi^\lambda_{G_1,k_1}(\xi_1) \cdots \Psi^\lambda_{G_m,k_m}(\xi_m) \\ &+ \sum_{\lambda \in \mathbb{N}_0 : 2^\lambda < 2^8 C_0 m\sqrt{n}} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\vec{k} \in \mathcal{U}} b^{\lambda,\gamma}_{\vec{G},\vec{k}} \Psi^\lambda_{G_1,k_1}(\xi_1) \cdots \Psi^\lambda_{G_m,k_m}(\xi_m) \\ =:& \ \sigma^{(1)}_\gamma(\vec{\xi}) + \sigma^{(2)}_\gamma(\vec{\xi}).\end{aligned}$$

We are only concerned with $\sigma^{(1)}_\gamma$ as a similar and simpler argument is applicable to the other one since the sum over λ in $\sigma^{(2)}_\gamma$ is finite sum.

If $2^8 C_0 m\sqrt{n} \leq 2^\lambda$, then $b^{\lambda,\gamma}_{\vec{G},\vec{k}}$ vanishes unless $2^{\lambda-2} \leq |\vec{k}| \leq 2^{\lambda+2}$ due to (48) and the compact support of $\Psi_{\vec{G}}$. Thus, letting

$$\mathcal{U}^\lambda := \{\vec{k} \in \mathcal{U} : 2^{\lambda-2} \leq |\vec{k}| \leq 2^{\lambda+2}\},$$

we write

$$\sigma^{(1)}_\gamma(\vec{\xi}) = \sum_{\lambda : 2^\lambda \geq 2^8 C_0 m\sqrt{n}} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\vec{k} \in \mathcal{U}^\lambda} b^{\lambda,\gamma}_{\vec{G},\vec{k}} \Psi^\lambda_{G_1,k_1}(\xi_1) \cdots \Psi^\lambda_{G_m,k_m}(\xi_m).$$

Now we split $\mathcal{U}^\lambda$ into m disjoint subsets

$$\begin{aligned}\mathcal{U}^\lambda_1 &:= \{\vec{k} \in \mathcal{U}^\lambda : |k_1| \geq 2C_0\sqrt{n} > |k_2| \geq \cdots \geq |k_m|\} \\ \mathcal{U}^\lambda_2 &:= \{\vec{k} \in \mathcal{U}^\lambda : |k_1| \geq |k_2| \geq 2C_0\sqrt{n} > |k_3| \geq \cdots \geq |k_m|\} \\ &\vdots \\ \mathcal{U}^\lambda_m &:= \{\vec{k} \in \mathcal{U}^\lambda : |k_1| \geq \cdots \geq |k_m| \geq 2C_0\sqrt{n}\}\end{aligned}$$

and accordingly,

$$\sigma^{(1)}_\gamma(\vec{\xi}) = \sum_{l=1}^m \sigma^{(1)}_{\gamma,l}(\vec{\xi})$$

where

$$\sigma^{(1)}_{\gamma,l}(\vec{\xi}) := \sum_{\lambda : 2^\lambda \geq 2^8 C_0 m\sqrt{n}} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\vec{k} \in \mathcal{U}^\lambda_l} b^{\lambda,\gamma}_{\vec{G},\vec{k}} \Psi^\lambda_{G_1,k_1}(\xi_1) \cdots \Psi^\lambda_{G_m,k_m}(\xi_m).$$

Then it is enough to show that for each $1 \leq l \leq m$

$$\begin{aligned}&\Big\| \sum_{\lambda : 2^\lambda \geq 2^8 C_0 m\sqrt{n}} \sum_{\vec{G} \in \mathcal{I}^\lambda} \sum_{\gamma \in \mathbb{Z}} \sum_{\vec{k} \in \mathcal{U}^\lambda_l} b^{\lambda,\gamma}_{\vec{G},\vec{k}} \Big(\prod_{j=1}^m L^{\lambda,\gamma}_{G_j,k_j} f_j \Big) \Big\|_{L^{2/m}} \\ &\lesssim \sup_{j \in \mathbb{Z}} \big\| \sigma(2^j \vec{\cdot}\,) \widehat{\Phi^{(m)}} \big\|_{L^q_s((\mathbb{R}^n)^m)} \prod_{j=1}^m \|f_j\|_{L^2} \qquad (51)\end{aligned}$$

where $L^{\lambda,\gamma}_{G,k}$ is defined as in (37).

Observe that if $|k| \geq 2C_0\sqrt{n}$ and $|2^{\lambda-\gamma}\xi - k| \leq C_0\sqrt{n}$, then

$$C_0\sqrt{n} \leq |k| - C_0\sqrt{n} \leq 2^{\lambda-\gamma}|\xi| \leq |k| + C_0\sqrt{n} \leq 2^{\lambda+2} + C_0\sqrt{n} \leq 2^{\lambda+3},$$

which implies

$$L^{\lambda,\gamma}_{G,k} f(x) = L^{\lambda,\gamma}_{G,k} f^{\lambda,\gamma}(x) \tag{52}$$

where $f^{\lambda,\gamma} := \big(\widehat{f}\chi_{C_0\sqrt{n}2^{\gamma-\lambda}\leq|\cdot|\leq 2^{\gamma+3}}\big)^{\vee}$. Furthermore, a direct computation with Plancherel's idendity proves

$$\Big(\sum_{\gamma\in\mathbb{Z}} \|f^{\lambda,\gamma}\|_{L^2}^2\Big)^{1/2} \lesssim_{C_0} (\lambda+3)^{1/2}\|f\|_{L^2}. \tag{53}$$

Let

$$\mathfrak{T}^{\lambda,\gamma}_{l,\vec{G}}(f_1,\dots,f_m)(x) := \sum_{\vec{k}\in\mathcal{U}^{\lambda}_l} b^{\lambda,\gamma}_{\vec{G},\vec{k}} \Big(\prod_{j=1}^{l} L^{\lambda,\gamma}_{G_j,k_j} f_j^{\lambda,\gamma}(x)\Big)\Big(\prod_{j=l+1}^{m} L^{\lambda,\gamma}_{G_j,k_j} f_j(x)\Big).$$

Then, due to (52), the left-hand side of (51) is less than

$$\bigg(\sum_{\lambda:2^{\lambda}\geq 2^8 C_0 m\sqrt{n}} \sum_{\vec{G}\in\mathcal{I}^{\lambda}} \Big\| \sum_{\gamma\in\mathbb{Z}} \mathfrak{T}^{\lambda,\gamma}_{l,\vec{G}}(f_1,\dots,f_m)\Big\|_{L^{2/m}}^{2/m}\bigg)^{m/2}. \tag{54}$$

We claim that for $1 \leq l \leq m$ there exists a constant $C > 0$ such that

$$\begin{aligned} \Big\| \sum_{\gamma\in\mathbb{Z}} \mathfrak{T}^{\lambda,\gamma}_{l,\vec{G}}(f_1,\dots,f_m)\Big\|_{L^{2/m}} &\leq C 2^{-\lambda(s-\max(\frac{(m-1)n}{2},\frac{mn}{q}))}(\lambda+3)^m \\ &\quad \times \sup_{j\in\mathbb{Z}} \big\|\sigma(2^j\vec{\cdot})\widehat{\Phi^{(m)}}\big\|_{L^q_s((\mathbb{R}^n)^m)} \prod_{j=1}^{m} \|f_j\|_{L^2}, \end{aligned} \tag{55}$$

which clearly implies that (54) is majorized by the right-hand side of (51) as

$$\bigg(\sum_{\lambda:2^{\lambda}\geq 2^8 C_0 m\sqrt{n}} 2^{-\frac{2\lambda}{m}(s-\max(\frac{(m-1)n}{2},\frac{mn}{q}))}(\lambda+3)^2\bigg)^{m/2} < \infty,$$

which is due to the assumption $s > \max\big(\frac{(m-1)n}{2},\frac{mn}{q}\big)$.

Therefore, let us prove (55).

7.1. **The case** $l = 1$**.** We utilize the Littlewood-Paley theory for Hardy spaces as in Section 6. There exists a unique polynomial $Q^{\lambda,\vec{G}}(x)$ such that

$$\begin{aligned} &\Big\| \sum_{\gamma\in\mathbb{Z}} \mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m) - Q^{\lambda,\vec{G}}\Big\|_{L^{2/m}} \\ &\quad \lesssim \Big\| \Big\{ \Phi_j^{(m)} * \Big(\sum_{\gamma\in\mathbb{Z}} \mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m)\Big)\Big\}_{j\in\mathbb{Z}}\Big\|_{L^{2/m}(\ell^2)}. \end{aligned} \tag{56}$$

Note that

$$2^{\lambda-3}\le 2^{\lambda-2}-2C_0\sqrt{mn}\le|\vec{k}|-(|k_2|^2+\cdots+|k_m|^2)^{1/2}\le|k_1|\le|\vec{k}|\le 2^{\lambda+2}$$

and this proves that

$$\mathrm{Supp}\big(\Psi^{\lambda}_{G_1,k_1}(\cdot/2^{\gamma})\big)\subset\{\xi\in\mathbb{R}^n:2^{\gamma-4}\le|\xi|\le 2^{\gamma+3}\}.$$

Moreover, since $|k_j|\le 2C_0\sqrt{n}$ for $2\le j\le m$,

$$\mathrm{Supp}\big(\Psi^{\lambda}_{G_j,k_j}(\cdot/2^{\gamma})\big)\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2^{-6}m^{-1}2^{\gamma}\}.$$

Therefore, the Fourier transform of $\mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m)$ is supported in the set $\{\xi\in\mathbb{R}^n:2^{\gamma-5}\le|\xi|\le 2^{\gamma+4}\}$ and the technique of (44) yields that the right-hand side of (56) is dominated by a constant times

$$\Big\|\Big(\sum_{\gamma\in\mathbb{Z}}\big|\mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m)\big|^2\Big)^{1/2}\Big\|_{L^{2/m}}.$$

The $L^{2/m}$-norm is bounded by

$$\sup_{\gamma\in\mathbb{Z}}\big\|\{b^{\lambda,\gamma}_{\vec{G},\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^\infty}2^{\lambda mn/2}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_1^{\lambda,\gamma}\|_{L^2}^2\Big)^{1/2}\prod_{j=2}^m\|f_j\|_{L^2}$$

thanks to Proposition 2.4. The embedding $\ell^q\hookrightarrow\ell^\infty$ and (50) imply

(57)
$$\sup_{\gamma\in\mathbb{Z}}\big\|\{b^{\lambda,\gamma}_{\vec{G},\vec{k}}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^\infty}\lesssim 2^{-\lambda(s-mn/q+mn/2)}\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\,\vec{\cdot}\,)\widehat{\Phi^{(m)}}\big\|_{L^q_s((\mathbb{R}^n)^m)}.$$

This, together with (53), finally proves that the left-hand side of (56) is dominated by a constant multiple of

$$2^{-\lambda(s-mn/q)}(\lambda+3)^{1/2}\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\,\vec{\cdot}\,)\widehat{\Phi^{(m)}}\big\|_{L^q_s((\mathbb{R}^n)^m)}\prod_{j=1}^m\|f_j\|_{L^2}$$

and accordingly,

$$\sum_{\gamma\in\mathbb{Z}}\mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m)-Q^{\lambda,\vec{G}}\in L^{2/m}.$$

Moreover, Proposition 2.4, together with (57), yields that

$$\Big\|\sum_{\gamma\in\mathbb{Z}}\mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m)\Big\|_{L^{2/m}}\le\Big\|\sum_{\gamma\in\mathbb{Z}}\big|\mathfrak{T}^{\lambda,\gamma}_{1,\vec{G}}(f_1,\dots,f_m)\big|\Big\|_{L^{2/m}}$$
$$\lesssim 2^{-\lambda(s-mn/q)}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_1^{\lambda,\gamma}\|_{L^2}\Big)\prod_{j=2}^m\|f_j\|_{L^2}$$

and, similarly to (42), we have

$$\|f_1^{\lambda,\gamma}\|_{L^2} = \left[\int_{C_0\sqrt{n}2^{\gamma-\lambda}\le|\xi|\le 2^{\gamma+3}} |\widehat{f_1}(\xi)|^2 d\xi\right]^{\frac{1}{2}} \lesssim_N \begin{cases} 2^{(\gamma+3)n/2}, & \gamma<0 \\ 2^{-(\gamma-\lambda)(N-n/2)} & \gamma\ge 0 \end{cases}$$

for sufficiently large $N > n/2$. Using the argument that led to (43), we see that

$$\sum_{\gamma\in\mathbb{Z}} \mathfrak{T}_{1,\vec{G}}^{\lambda,\gamma}(f_1,\dots,f_m) \in L^{2/m}$$

and thus $Q^{\lambda,\vec{G}} = 0$. Then the inequality (55) for $l=1$ follows.

7.2. **The case** $2\le l\le m$**.** If $0<q<\frac{2l}{l-1}$, we simply apply Proposition 2.4 to have

$$\begin{aligned}&\Big\|\sum_{\gamma\in\mathbb{Z}} \mathfrak{T}_{l,\vec{G}}^{\lambda,\gamma}(f_1,\dots,f_m)\Big\|_{L^{2/m}} \\ &\lesssim \sup_{\gamma\in\mathbb{Z}} \big\|\{b_{\vec{G},\vec{k}}^{\lambda,\gamma}\}_{\vec{k}\in(\mathbb{Z}^n)^m}\big\|_{\ell^q} 2^{\lambda mn/2}\Big[\prod_{j=1}^{l}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_j^{\lambda,\gamma}\|_{L^2}^2\Big)^{1/2}\Big]\Big[\prod_{j=l+1}^{m}\|f_j\|_{L^2}\Big]\end{aligned}$$

where the embedding $\ell^q\hookrightarrow\ell^\infty$ is applied. Then the last expression is no more than a constant multiple of

$$2^{-\lambda(s-mn/q)}(\lambda+3)^{l/2}\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\vec{\cdot}\,)\widehat{\Phi^{(m)}}\big\|_{L_s^q((\mathbb{R}^n)^m)}\prod_{j=1}^{m}\|f_j\|_{L^2}$$

by using (50) and (53). Then (55) follows.

If $\frac{2l}{l-1}\le q<\infty$, applying the third statement of Proposition 2.4, we obtain

$$\begin{aligned}&\Big\|\sum_{\gamma\in\mathbb{Z}} \mathfrak{T}_{l,\vec{G}}^{\lambda,\gamma}(f_1,\dots,f_m)\Big\|_{L^{2/m}} \\ &\lesssim E_{q,l,\lambda}2^{-\lambda(s-mn/q+mn/2)}2^{\lambda mn/2}\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\vec{\cdot}\,)\widehat{\Phi^{(m)}}\big\|_{L_s^q((\mathbb{R}^n)^m)} \\ &\qquad\qquad \times\prod_{j=1}^{l}\Big(\sum_{\gamma\in\mathbb{Z}}\|f_j^{\lambda,\gamma}\|_{L^2}^2\Big)^{1/2}\Big(\prod_{j=l+1}^{m}\|f_j\|_{L^2}\Big)\end{aligned}$$

where

$$E_{q,l,\lambda} := \begin{cases}\lambda^{l/2}, & q=\frac{2l}{l-1} \\ 2^{\lambda n(l/2-l/q-1/2)}, & q>\frac{2l}{l-1}\end{cases}.$$

Noticing that $\Big(\sum_{\gamma\in\mathbb{Z}}\|f_j^{\lambda,\gamma}\|_{L^2}^2\Big)^{1/2}\lesssim(\lambda+3)^{1/2}\|f_j\|_{L^2}$, we finally obtain that

$$\Big\|\sum_{\gamma\in\mathbb{Z}} \mathfrak{T}_{l,\vec{G}}^{\lambda,\gamma}(f_1,\dots,f_m)\Big\|_{L^{2/m}} \lesssim F_{q,l,\lambda}^{(s,m,n)}\sup_{j\in\mathbb{Z}}\big\|\sigma(2^j\vec{\cdot}\,)\widehat{\Phi^{(m)}}\big\|_{L_s^q((\mathbb{R}^n)^m)}\prod_{j=1}^{m}\|f_j\|_{L^2}$$

where
$$F^{(s,m,n)}_{q,l,\lambda} := E_{q,l,\lambda} 2^{-\lambda(s-mn/q)} (\lambda+3)^{l/2}.$$
It is easy to check that for $2 \le l \le m$ and $\frac{2l}{l-1} \le q$
$$F^{(s,m,n)}_{q,l,\lambda} \lesssim 2^{-\lambda(s-\max(\frac{(m-1)n}{2}, \frac{mn}{q}))} (\lambda+3)^m$$
and the proof of (55) is complete.

8. CONCLUDING REMARKS

In this article we focused on the $L^2 \times \cdots \times L^2 \to L^{2/m}$ boundedness for several fundamental m-linear operators. In future work we plan to obtain similar initial estimates for maximal singular integrals and maximal multipliers.

The $L^2 \times \cdots \times L^2$ estimates obtained in this paper provide crucial initial bounds that provide the cornerstone needed to launch a complete boundedness study on general products of Lebesgue spaces. Certainly our initial estimates can be extended to include points obtained by duality and interpolation; these are called local L^2 points. For the remaining points there are techniques available, for instance, interpolation between dyadic pieces of an operator between *good* local L^2 points and *bad* points near the boundary of the region $1 < p_1, \dots, p_m < \infty$, $1/m < p < \infty$; this technique was developed in [17] in the bilinear case. We chose not to pursue this line of investigation here in order to direct our focus on the idea of wavelet expansions and shorten the exposition. We plan to pursue general $L^{p_1} \times \cdots \times L^{p_m} \to L^p$ boundedness for many multilinear operators in subsequent work. It should be mentioned that in a recent manuscript of Heo, Lee, Hong, Yang, Lee, and Park [25] the extension to the full range of indices was obtained for Theorem 1.3, when $q = 2$, although the case of general q remains unresolved.

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REFERENCES

[1] E. Buriánková, L. Grafakos, D. He, and P. Honzík, *The lattice bump multiplier problem*, Studia Math. **262** (2022), 225-240.
[2] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution,* II. Adv. in Math. **24** (1977), no. 1, 101–171.
[3] A.P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952) 85–139.
[4] R. R. Coifman and Y. Meyer, *Commutateurs d' intégrales singulières et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[5] R. R. Coifman and Y. Meyer, *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[6] R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977) 569-645.

[7] J. M. Conde-Alonso, A. Culiuc, F. Di Plinio, and Y. Ou, *A sparse domination principle for rough singular integrals*, Anal. PDE **10** (2017), 1255–1284.

[8] I. Daubechies, *Orthonormal bases of compactly supported wavelets*, Comm. Pure Appl. Math. **41** (1988) 909-996.

[9] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1992).

[10] F. Di Plinio, K. Li, H. Martikainen, and E. Vuorinen, *Multilinear operator-valued Calderón-Zygmund theory*, J. Funct. Anal. **279** (2020), 62 pp.

[11] F. Di Plinio, K. Li, H. Martikainen, and E. Vuorinen, *Multilinear singular integrals on non-commutative L^p spaces*, Math. Ann. **378** (2020), 1371–1414.

[12] F. Di Plinio, B. Wick, and T. Williams, *Wavelet representation of singular integral operators*, arXiv preprint arXiv:2009.01212.

[13] J. Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, **29**, American Mathematical Society, Providence, RI, 2001.

[14] J. Duoandikoetxea and J.-L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986) 541-561.

[15] T. Hytönen and S. Lappas, *Dyadic representation theorem using smooth wavelets with compact support*, arXiv preprint arXiv:2003.04019.

[16] L. Grafakos, *Modern Fourier Analysis.,* 3rd Ed., GTM **250**, Springer NY, 2014.

[17] L. Grafakos, D. He, and P. Honzík, *Rough bilinear singular integrals*, Adv. Math. **326** (2018) 54-78.

[18] L. Grafakos, D. He, and P. Honzík, *The Hörmander multiplier theorem, II : The bilinear local L^2 case*, Math. Zeit. **289** (2018) 875-887.

[19] L. Grafakos, D. He, and L. Slavíková, $L^2 \times L^2 \to L^1$ *boundedness criteria*, Math. Ann. **376** (2020) 431-455.

[20] L. Grafakos, D. He, and L. Slavíková, *Failure of the Hörmander kernel condition for multilinear Calderón-Zygmund operators*, Comptes Rendus Acad. Sci. Math. **357** (2019), 382–388.

[21] L. Grafakos and H.V. Nguyen, *The Hörmander multiplier theorem, III : the complete bilinear case via interpolation*, Monatsh. Math. **190** (2019) 735-753.

[22] L. Grafakos, Z. Si, *The Hörmander multiplier theorem for multilinear operators*, J. Reine Angew. Math. **668** (2012), 133–147.

[23] L. Grafakos and R. H. Torres, *On multilinear singular integrals of Calderón-Zygmund type*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations [El Escorial, 2000], pp. 57–91, Publicacions Matemàtiques 2002, Vol. Extra.

[24] L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory,* Adv. in Math. **165** (2002), 124–164.

[25] J. Lee, Y. Heo, S. Hong, J.B. Lee, B. Park, Y. Park, and C. Yang, *The Hörmander multiplier theorem for n-linear operators*, Math. Ann. **381** (2021), 499-555.

[26] T. Kato, A. Miyachi, and N. Tomita, *Boundedness of bilinear pseudo-differential operators of $S_{0,0}$-type on $L^2 \times L^2$,* J. Pseudo-Differ. Oper. Appl. **12** (2021) Paper No. 15, 38pp.

[27] T. Kato, A. Miyachi, and N. Tomita, *Boundedness of multilinear pseudo-differential operators of $S_{0,0}$-type in L^2-based amalgam spaces,* J. Math. Soc. Japan. Published online.

[28] M. T. Lacey and C. M. Thiele, *L^p bounds for the bilinear Hilbert transform,* $2 < p < \infty$, Ann. Math. **146** (1997), 693–724.

[29] M. T. Lacey and C. M. Thiele, *On Calderón's conjecture*, Ann. Math. **149** (1999), 475–496.

[30] Y. Meyer, *Wavelets and operators*, Cambridge Studies in Advanced Mathematics, vol. 37. Cambridge University Press, Cambridge, 1992.

[31] Y. Meyer and R. R. Coifman, *Wavelets, Calderón-Zygmund Operators and Multilinear Operators*, Cambridge Studies in Advanced Mathematics, **48**, Cambridge University Press, Cambridge, 1997.

[32] A. Miyachi and N. Tomita, *Minimal smoothness conditions for bilinear Fourier multipliers*, Rev. Mat. Iberoamer. **29** (2013) 495-530.

[33] E. Palsson and S. Sovine, *The triangle averaging operator*, J. Funct. Anal. **279** (2020), no. 8, 108671, 21 pp.

[34] N. Tomita, *A Hörmander type multiplier theorem for multilinear operators*, J. Funct. Anal. **259** (2010) 2028–2044.

[35] H. Triebel, *Theory of function spaces. III*, Monographs in Mathematics, vol. 100, Birkhäuser, Basel-Boston-Stuttgart, 2006.

[36] H. Triebel, *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*, EMS Tracts in Mathematics, vol. 11, European Mathematical Society (EMS), Zürich, 2010.

[37] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., **33** (1986) 131-174.

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